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A GENERATING FUNCTION FOR ASSOCIATED LEGENDRE POLYNOMIALS

By F. BRAFMAN (*Southern Illinois University*)

[Received 25 May 1956; in revised form 1 November 1956]

1. Introduction

JACOBI polynomials may be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\ 1+\alpha+\beta+n;\\ 1+\alpha;\end{matrix}\ \tfrac{1}{2}(1-x)\right]. \tag{1}$$

The Legendre polynomials result from putting $\alpha = \beta = 0$ and are commonly written without the superscript notation as

$$P_n(x) = P_n^{(0,0)}(x). \tag{2}$$

Let $$\rho = (1-2xt+t^2)^{\frac{1}{2}}, \quad \text{where } \rho = 1 \text{ for } t = 0, \tag{3}$$

$$\phi = (\rho^2+2tux-2t^2u+u^2t^2)^{\frac{1}{2}}, \quad \text{where } \phi = 1 \text{ for } t = 0. \tag{4}$$

Then the main result of this note will be

$$\rho^{-1}\left(\frac{1+t+\rho}{1-t+\rho}\right)^{\alpha} \frac{k!\,k!}{(1+\alpha)_k(1-\alpha)_k} P_k^{(\alpha,-\alpha)}\left(\frac{\phi-tu}{\rho}\right) P_k^{(-\alpha,\alpha)}\left(\frac{\phi+tu}{\rho}\right)$$
$$= \sum_{n=0}^{\infty} P_n^{(\alpha,-\alpha)}(x)\,{}_3F_2\left[\begin{matrix}-k,\ k+1,\ -n;\\ 1+\alpha,\ 1-\alpha;\end{matrix}\ u\right]t^n, \tag{5}$$

valid in a neighbourhood of $t = 0$. This will generalize a result obtained by Rice [(1) 115, (2.14)]. Rice's result corresponds to (5) with $\alpha = 0$, and was obtained by generalizing a result of Bateman's.

Equation (5) can be presented in the alternative form

$$\rho^{-1}\left(\frac{1+t+\rho}{1-t+\rho}\right)^{\alpha} {}_2F_1\left[\begin{matrix}-k,\ 1+k;\\ 1+\alpha;\end{matrix}\ \frac{\rho-\phi+tu}{2\rho}\right] {}_2F_1\left[\begin{matrix}-k,\ 1+k;\\ 1-\alpha;\end{matrix}\ \frac{\rho-\phi-tu}{2\rho}\right]$$
$$= \sum_{n=0}^{\infty} P_n^{(\alpha,-\alpha)}(x)\,{}_3F_2\left[\begin{matrix}-k,\ 1+k,\ -n;\\ 1+\alpha,\ 1-\alpha;\end{matrix}\ u\right]t^n. \tag{6}$$

The associated Legendre polynomials can be defined [(2) 326] by

$$P_n^m(x) = \frac{1}{\Gamma(1-m)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} {}_2F_1\left[\begin{matrix}-n,\ 1+n;\\ 1-m;\end{matrix}\ \tfrac{1}{2}(1-x)\right], \tag{7}$$

with the usual limiting definition when m is a positive integer. Thus

$$P_n^{(\alpha,-\alpha)}(x) = \Gamma(1+\alpha)\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}\alpha} \frac{(1+\alpha)_n}{n!} P_n^{-\alpha}(x). \tag{8}$$

Hence equation (5) or (6) may be interpreted as a generating function for associated Legendre polynomials by substitution from (8).

The author desires to thank Dr. T. W. Chaundy for suggestions which resulted in great simplifications in the proof below.

2. Proof of (6)

Denote the left-hand side of (6) by A and the right-hand side by B. Then, by use of Bailey's reduction formula on the Appell function F_4 [(**3**), 81, (1)],

$$A = \rho^{-1}\left[\frac{1+t+\rho}{1-t+\rho}\right]^{\alpha} F_4\left[\begin{matrix}-k,\ 1+k; \\ 1+\alpha,\ 1-\alpha;\end{matrix}\ \frac{tu(t-x+\rho)}{2\rho^2},\ \frac{tu(t-x-\rho)}{2\rho^2}\right]$$

$$= \rho^{-1}\left[\frac{1+t+\rho}{1-t+\rho}\right]^{\alpha} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-k)_{m+n}(1+k)_{m+n}(tu)^{m+n}(t-x+\rho)^m(t-x-\rho)^n}{(1+\alpha)_m(1-\alpha)_n\, m!\, n!\, (2\rho^2)^{m+n}}$$

$$= \rho^{-1}\left[\frac{1+t+\rho}{1-t+\rho}\right]^{\alpha} \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{(-k)_m(1+k)_m(tu)^m(t-x+\rho)^m}{(1+\alpha)_{m-n}(1-\alpha)_n(m-n)!\, n!(2\rho^2)^m}\left[\frac{t-x-\rho}{t-x+\rho}\right]^n$$

$$= \rho^{-1}\left[\frac{1+t+\rho}{1-t+\rho}\right]^{\alpha} \sum_{m=0}^{\infty} \frac{(-k)_m(1+k)_m(tu)^m(t-x+\rho)^m}{m!\,(1+\alpha)_m(2\rho^2)^m}\times$$

$$\times\, {}_2F_1\left[\begin{matrix}-m,\ -\alpha-m; \\ 1-\alpha;\end{matrix}\ \frac{t-x-\rho}{t-x+\rho}\right]$$

$$= \rho^{-1}\left[\frac{1+t+\rho}{1-t+\rho}\right]^{\alpha} \sum_{m=0}^{\infty} \frac{(-k)_m(1+k)_m\, u^m}{(1+\alpha)_m(1-\alpha)_m}\left(\frac{-t}{\rho}\right)^m P_m^{(\alpha,-\alpha)}\left(\frac{x-t}{\rho}\right). \tag{9}$$

On the right-hand side of (6), expand the ${}_3F_2$ and interchange summations. The interchange is easily justified by the uniform convergence of the series involved. The result is

$$B = \sum_{m=0}^{\infty} \frac{(-k)_m(1+k)_m\, u^m}{(1+\alpha)_m(1-\alpha)_m\, m!} \sum_{n=0}^{\infty} (-n)_m\, P_n^{(\alpha,-\alpha)}(x)t^n. \tag{10}$$

Thus
$$A-B = \sum_{m=0}^{\infty} \frac{(-k)_m(1+k)_m\, u^m}{(1+\alpha)_m(1-\alpha)_m} R_m, \tag{11}$$

where

$$R_m = \rho^{-1}\left[\frac{1+t+\rho}{1-t+\rho}\right]^{\alpha}\left(\frac{-t}{\rho}\right)^m P_m^{(\alpha,-\alpha)}\left(\frac{x-t}{\rho}\right) - \frac{1}{m!}\sum_{n=0}^{\infty}(-n)_m\, P_n^{(\alpha,-\alpha)}(x)t^n. \tag{12}$$

Then a necessary and sufficient condition that $A = B$ is that R_m should vanish for all m. Hence it is necessary and sufficient that

$$C = \sum_{m=0}^{\infty} R_m v^m \tag{13}$$

should vanish identically in v. From (12) it follows that

$$C = \rho^{-1}\left[\frac{1+t+\rho}{1-t+\rho}\right]^{\alpha} \sum_{m=0}^{\infty} \left(\frac{-vt}{\rho}\right)^m P_m^{(\alpha,-\alpha)}\left(\frac{x-t}{\rho}\right) - \sum_{n=0}^{\infty} P_n^{(\alpha,-\alpha)}(x)\{t(1-v)\}^n. \tag{14}$$

This operation again involves an interchange of summations in the second term, but the factor $(-n)_m$ reduces the outer sum to a finite one, and so no justification is needed.

Let

$$X = (x-t)/\rho, \qquad T = -vt/\rho, \qquad R^2 = 1-2XT+T^2, \tag{15}$$

$$\tau = t(1-v), \qquad P^2 = 1-2x\tau+\tau^2,$$

with $R = P = 1$ at $t = 0$.

Then the application of a standard generating function for Jacobi polynomials [(4), 68, (4.4.5)] yields

$$C = \rho^{-1}\left[\frac{1+t+\rho}{1-t+\rho}\right]^{\alpha} R^{-1}\left[\frac{1+T+R}{1-T+R}\right]^{\alpha} - P^{-1}\left[\frac{1+\tau+P}{1-\tau+P}\right]^{\alpha}. \tag{16}$$

Now
$$\rho R = P. \tag{17}$$

Hence
$$C = P^{-1}\left[\left(\frac{1+t+\rho}{1-t+\rho}\,\frac{\rho(1+T)+P}{\rho(1-T)+P}\right)^{\alpha} - \left(\frac{1+\tau+P}{1-\tau+P}\right)^{\alpha}\right]. \tag{18}$$

Again

$$\frac{1+t+\rho}{1-t+\rho} = \frac{x-\rho-t}{x-1}, \quad \frac{1+\tau+P}{1-\tau+P} = \frac{x-P-\tau}{x-1}, \quad \frac{\rho(1+T)+P}{\rho(1-T)+P} = \frac{x-P-\tau}{x-\rho-t}.$$

Thus $C = 0$ in (18), and (6) is proved.

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FOURIER SERIES WITH GAPS (II)

By P. B. KENNEDY (*Cork*)

[Received 1 August 1956]

1. SUPPOSE that $f(x) \in L(-\pi, \pi)$ and has period 2π, and that the Fourier coefficients a_n, b_n of $f(x)$ vanish except possibly when n is a member of a given strictly increasing sequence of positive integers $\{n_k\}$ satisfying

$$n_{k+1}-n_k \to \infty \quad (k \to \infty). \tag{1.1}$$

In a previous note (**2**) it was shown that, *if* $0 < \alpha < 1$, *and* $f(x) \in \mathrm{Lip}\,\alpha$ *in some interval* $|x-x_0| \leqslant \delta$, *then*

$$a_n, b_n = O(n^{-\alpha}) \quad (n \to \infty); \tag{1.2}$$

and, if $\alpha > \frac{1}{2}$, *then*

$$\sum (|a_n|+|b_n|) < \infty. \tag{1.3}$$

With $\delta = \pi$ and without the hypothesis (1.1), this theorem is classical [(**4**), 18, 135]; and, with a more restrictive gap condition than (1.1), this and other similar theorems were first obtained by Noble (**3**).

From (1.1) it follows that

$$n_k/k \to \infty, \tag{1.4}$$

but the converse is false. Certain theorems similar to those in (**2**) are known to hold under the hypothesis (1.4): for example, if $f(x)$ vanishes throughout an interval, then (1.4) implies the vanishing of $f(x)$ almost everywhere [(**1**), 237]. It is therefore natural to ask whether (1.1) can be replaced by (1.4) in the theorem quoted above. In this note I show, in answer to this question, that neither (1.2) nor (1.3) remains true in general under the gap hypothesis (1.4), or even under a much stronger hypothesis of a similar kind. I prove, in fact, the following theorem.

THEOREM. *Let* $0 < \eta < \pi$, *and let* $\phi(t) \to \infty$ *steadily as* $t \to \infty$. *Then there exist a strictly increasing sequence of positive integers* $\{n_k\}$ *and a function* $f(x) \in L^2(-\pi, \pi)$ *such that*

(*a*) $\liminf_{k\to\infty} n_k \exp\left\{-\frac{k(\pi-\eta)}{22}\right\} \geqslant 1$;

(*b*) *the Fourier coefficients* a_n, b_n *of* $f(x)$ *vanish except possibly when* $n = n_k$;

(*c*) *for every* $\alpha < 1$, $f(x) \in \mathrm{Lip}\,\alpha$ *in* $(-\eta, \eta)$,

but

$$\limsup_{n\to\infty} (|a_n|+|b_n|)\phi(n) = \infty$$

and

$$\sum (|a_n|+|b_n|) = \infty.$$

Quart. J. Math. Oxford (2), 8 (1957), 84-88.

By taking $\phi(t) = t^{\alpha}$, where $0 < \alpha < 1$, we obtain from (*c*) that, in particular, (1.2) is false for the function $f(x)$ in the theorem.

2. In this section, $A_1, \ldots, A_4$ denote positive numbers depending on δ only. We need the following lemma, due to Noble (**3**).

LEMMA. *If $0 < \delta < \pi$, and m is a positive integer, there exists a trigonometrical polynomial*

$$T_m(x) = 1 + \sum_{j=1}^{m} t_j(m) \cos jx,$$

satisfying

(i) $$|T_m(x)| < A_1 \quad \textit{for all } x;$$

(ii) $$|T_m(x)| < A_2 m^2 \exp\left(-\frac{\delta m}{4e}\right) \quad (\delta \leqslant |x| \leqslant \pi);$$

(iii) $$|T'_m(x)| < A_1 m \quad \textit{for all } x;$$

(iv) $$\sum_{j=1}^{m} t_j^2(m) < A_3.$$

Noble does not explicitly state (iv) or the fact that $T_m(x)$ contains no sine terms; but (iv) follows at once from (i) and Bessel's inequality, and $T_m(x)$ is a partial sum of the Fourier series of an even function and so contains only cosine terms. Further, Noble states a less precise form of (ii), but his proof gives the above version.†

To prove the theorem, put $\delta = \pi - \eta$, and choose constants $\lambda > 1$ and μ $(0 < \mu < 1)$. Suitable values will be assigned to λ and μ later. Let $\{m_p\}$ be a sequence of positive integers satisfying

$$m_{p+1} > \lambda m_p, \tag{2.1}$$

$$p^{-1}\phi\left(\exp\frac{\lambda\mu\delta m_p}{\lambda - 1}\right) \to \infty \quad (p \to \infty).$$

It is plain that such a sequence exists since $\phi(t) \to \infty$ as $t \to \infty$. Put

$$M_p = 1 + \left[\exp\frac{\lambda\mu\delta m_p}{\lambda - 1}\right] \tag{2.2}$$

where, as usual, the square brackets denote the integer part. These definitions imply

$$M_p + m_p = o(M_{p+1} + m_{p+1}), \tag{2.3}$$

$$p^{-1}\phi(M_p) \to \infty. \tag{2.4}$$

† In (2.5) of (**3**), n^{1-m} should be n^{2-m}. I am indebted to the referee for this remark.

Moreover we suppose that m_1 is chosen so large that

$$M_1 - m_1 > 0, \qquad M_{p+1} - m_{p+1} > M_p + m_p \quad (p \geqslant 1). \tag{2.5}$$

Such a choice of m_1 is possible by (2.1) and (2.2).

Consider the series
$$\sum_{p=1}^{\infty} \frac{1}{p} T_{m_p}(x) \cos M_p x, \tag{2.6}$$

where $T_{m_p}(x)$ is as in the lemma. By (2.5) and the definition of $T_{m_p}(x)$, it is clear that the pth partial sum of (2.6) is the (M_p+m_p)th partial sum of a cosine series

$$\sum_1^{\infty} \alpha_n \cos nx, \tag{2.7}$$

where, by (iv) of the lemma,

$$\sum_1^{\infty} \alpha_n^2 = \sum_{p=1}^{\infty} \frac{1}{p^2}\Big\{1 + \tfrac{1}{2} \sum_{j=1}^{m_p} t_j^2(m_p)\Big\} < A_4.$$

By the Riesz–Fischer theorem [(4), 74], the last statement implies that (2.7) is the Fourier series of a function of $L^2(-\pi, \pi)$, say $g(x)$. Therefore a theorem due to Kolmogoroff [(4), 251] enables us to assert that, if the sequence of positive integers $\{\nu_p\}$ satisfies

$$\liminf_{p \to \infty} \frac{\nu_{p+1}}{\nu_p} > 1,$$

then
$$\sum_1^{\nu_p} \alpha_n \cos nx \to g(x) \quad (p \to \infty)$$

almost everywhere in $(-\pi, \pi)$. By (2.3) we may take $\nu_p = M_p + m_p$, and, in view of the relation between (2.6) and (2.7), Kolmogoroff's theorem then gives

$$g(x) = \sum_{p=1}^{\infty} \frac{1}{p} T_{m_p}(x) \cos M_p x \tag{2.8}$$

for almost all x in $(-\pi, \pi)$. In fact we may suppose (2.8) to hold *everywhere* in $\delta \leqslant |x| \leqslant \pi$ since the series is absolutely and uniformly convergent there, by (ii) of the lemma and (2.1).

Our next object is to show that, if the constants λ and μ are suitably chosen, $g(x) \in \text{Lip}\,\alpha$ in $\delta \leqslant |x| \leqslant \pi$ for every $\alpha < 1$. Suppose that x and $x+h$ both lie in $(-\pi, -\delta)$ or in (δ, π), and that $|h|$ is so small that there is an integer $q > 1$ satisfying

$$\exp\Big(-\frac{\delta m_q}{4e}\Big) < |h| \leqslant \exp\Big(-\frac{\delta m_{q-1}}{4e}\Big). \tag{2.9}$$

Then, by (ii) and (iii) of the lemma,

$$|g(x+h)-g(x)| \leqslant \sum_{1}^{q-1} \frac{1}{p}\left|\int_{x}^{x+h} \frac{d}{d\xi}\{T_{m_p}(\xi)\cos M_p \xi\}\, d\xi\right| +$$

$$+ \sum_{q}^{\infty} \frac{1}{p}\{|T_{m_p}(x+h)|+|T_{m_p}(x)|\}$$

$$< A_2|h| \sum_{1}^{q-1} \frac{M_p m_p^2}{p}\exp\left(-\frac{\delta m_p}{4e}\right)+A_1|h| \sum_{1}^{q-1} \frac{m_p}{p}+$$

$$+2A_2 \sum_{q}^{\infty} \frac{m_p^2}{p}\exp\left(-\frac{\delta m_p}{4e}\right).$$

By (2.1) and (2.9) the second and third terms are

$$O\left\{|h|m_{q-1}+m_q^2\exp\left(-\frac{\delta m_q}{4e}\right)\right\} = O(|h|^\alpha)$$

for every $\alpha < 1$. By (2.2) the first term is

$$O(|h|) \sum_{1}^{q-1} \frac{m_p^2}{p}\exp\left\{\left(\frac{\lambda\mu}{\lambda-1}-\frac{1}{4e}\right)\delta m_p\right\},$$

and, by (2.1), this is $O(|h|)$ if

$$\frac{\lambda\mu}{\lambda-1} < \frac{1}{4e}. \tag{2.10}$$

Thus, if (2.10) is true, $g(x+h)-g(x) = O(|h|^\alpha)$ for every $\alpha < 1$, as required.

Next, since $\alpha_n = p^{-1}$ when $n = M_p$, we have

$$\limsup_{n\to\infty} |\alpha_n|\phi(n) = \infty \tag{2.11}$$

by (2.4), and

$$\sum |\alpha_n| = \infty. \tag{2.12}$$

Also $\alpha_n = 0$ when

$$M_{p-1}+m_{p-1} < n < M_p-m_p.$$

If $\{n_k\}$ is the sequence of values of n for which $\alpha_n \neq 0$, then for every k there exists $p = p(k)$ such that

$$M_p-m_p \leqslant n_k \leqslant M_p+m_p,$$

and, by (2.1),

$$k \leqslant 2 \sum_{j=1}^{p} m_j < \frac{2\lambda}{\lambda-1} m_p.$$

Thus $\displaystyle \liminf_{k\to\infty} n_k \exp(-\tfrac{1}{2}\mu\delta k) \geqslant \lim_{p\to\infty} (M_p-m_p)\exp\left(-\frac{\lambda\mu\delta m_p}{\lambda-1}\right) = 1$

by (2.2). Hence, if we put $\mu = \frac{1}{11}$, assertion (*a*) of the theorem is true since $\delta = \pi - \eta$; and, since $4e < 11$, (2.10) is then satisfied if λ is sufficiently large.

The function $f(x)$ of the theorem is obtained by defining $g(x)$ outside $(-\pi, \pi)$ so as to have period 2π, and putting $f(x) = g(x+\pi)$. For then $f(x) \in \text{Lip}\,\alpha$ in $(-\pi+\delta, \pi-\delta)$, i.e. in $(-\eta, \eta)$, for every $\alpha < 1$. Moreover the Fourier series of $f(x)$ is

$$\sum_{1}^{\infty} (-1)^n \alpha_n \cos nx,$$

and so the remaining assertions of the theorem follow from (2.11) and (2.12). This proves the theorem.

3. Let the sequences $\{m_p\}$, $\{M_p\}$ be as in § 2. Then the function $f(x)$ defined by

$$f(x-\pi) = \sum_{p=1}^{\infty} \frac{1}{m_p^2} T_{m_p}(x) \cos M_p x$$

is continuous everywhere and belongs to the class Lip 1 in $(-\eta, \eta)$. In particular, $f(x)$ has bounded variation in $(-\eta, \eta)$. The nth Fourier coefficients of $f(x)$ are not $O(n^{-1})$ although they vanish except possibly when $n = n_k$. This example shows that we cannot replace (1.1) by the hypothesis (*a*) in Theorem V (i) of (**2**). I am unable to decide whether Theorem V (iv) of (**2**) survives under the hypothesis (1.4), or under a stronger hypothesis of the same kind such as (*a*).

I wish to thank the referee for his suggestions.

Note added in proof, 8 March 1957. In Theorem V (iv) of (**2**), (1.1) cannot be replaced by the hypothesis (*a*) of the theorem proved above. For, with the notation of § 2 a counter example is provided by

$$\sum_{p=1}^{\infty} p^{-1} M_p S_p(x), \quad \text{where} \quad S_p(x) = \int_{-\pi}^{x} T_{m_p}(\xi) \cos M_p \xi \, d\xi.$$

This, when each $S_p(x)$ is written *in extenso*, is the Fourier series of a function of $L^2(-\pi, \pi)$ having a continuous derivative in $(-\pi, -\delta)$.

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GAUGE-INVARIANT GENERALIZATION OF FIELD THEORIES WITH ASYMMETRIC FUNDAMENTAL TENSOR

By H. A. BUCHDAHL (*Hobart*)

[Received 13 August 1956]

1. (*a*) The General Theory of relativity in its original form is based—with regard only to its mathematical aspects—upon a quadratic differential form (the 'metric') whose coefficients (the 'gravitational potentials') are, by hypothesis, the components of a symmetric covariant tensor g_{ik} $(i,\ k = 1,\ldots,\ 4)$. The field equations are then certain covariant sets of equations which involve the g_{ik} and their first and second derivatives only. As it stands this mathematical apparatus is evidently unable to accommodate electromagnetism. To do so a generalization of it will be required which (presumably) involves in a 'natural' way not only a symmetric covariant tensor of valence 2, but also either (i) a covariant vector, to play the part of electromagnetic potential, or (ii) a bi-vector, to play the part of electromagnetic field tensor. In fact, of the attempted generalizations which retain the dimensional number four, that is to say which operate solely with four independent variables, the best known are perhaps those of Weyl (**1**) and of Einstein (**2**)—historically the first and last to date—and these respectively represent just the alternatives (i) and (ii) mentioned above. It is appropriate to review very briefly the analytical apparatus of these two theories, since they are to be united into a whole in the present paper.

(*b*) Weyl's theory involves as basic field quantities a *symmetrical* covariant tensor g_{ik} and a covariant vector k_i. (Except where otherwise indicated all indices will be allowed to run from 1 to n instead of from 1 to 4 only.) The linear connexion of a Weyl space W_n is *symmetrical* and is defined as†

$$\Gamma^s_{ik} = \{ik, s\} - (\delta^s_{(i} k_{k)} - \tfrac{1}{2} g_{ik} k^s), \tag{1.1}$$

where the Christoffel symbols $\{ik, s\}$ and the processes of raising and lowering indices refer to g_{ik} and its reciprocal. The Γ^s_{ik} are *gauge-invariant*, i.e. in the gauge transformation

$$g_{ik} \to \lambda g_{ik}, \qquad k_i \to k_i + (\log \lambda)_{,i}, \tag{1.2}$$

† To indicate mixing and alternating over indices Schouten's general notation (**3**) will be used side by side with Einstein's more specialized notation which is useful only for adjacent pairs of indices.

Quart. J. Math. Oxford (2), 8 (1957), 89-96

where λ is an arbitrary function of the coordinates, the Γ^s_{ik} transform into themselves. In this theory only gauge-covariant tensors and tensor-densities are considered, i.e. those which in the gauge transformation (1.2) merely multiply themselves by a power of λ. Explicitly, $\mathfrak{T}^{i_1\ldots i_\mu}_{k_1\ldots k_\nu}$ will be said to be a *tensor-density of coordinate-weight* c and *gauge-weight* v if in a coordinate transformation it multiplies itself by j^c and in a gauge transformation by λ^v, j being the Jacobian of the coordinate transformation. From $\mathfrak{T}^{i_1\ldots i_\mu}_{k_1\ldots k_\nu}$ a new gauge-covariant tensor-density possessing the *same* gauge-weight may be induced by means of the process of gauge-invariant covariant differentiation defined by (4)†

$$\mathfrak{T}^{i_1\ldots i_\mu}_{k_1\ldots k_\nu;l} = \mathfrak{T}^{i_1\ldots i_\mu}_{k_1\ldots k_\nu,l} - \sum_{\lambda=1}^{\nu} \Gamma^s_{k_\lambda l}\,\mathfrak{T}^{i_1\ldots i_\mu}_{k_1\ldots k_{\lambda-1}sk_{\lambda+1}\ldots k_\nu} +$$

$$+\sum_{\lambda=1}^{\mu} \Gamma^{i_\lambda}_{sl}\,\mathfrak{T}^{i_1\ldots i_{\lambda-1}si_{\lambda+1}\ldots i_\mu}_{k_1\ldots k_\nu} - (c\Gamma^s_{\underline{ls}}+vk_l)\mathfrak{T}^{i_1\ldots i_\mu}_{k_1\ldots k_\nu}. \tag{1.3}$$

Subscripts following a comma and semicolon denote as usual ordinary and covariant differentiation respectively. The bar under the two subscripts of the $\Gamma^s_{\underline{ls}}$ which appears in the last term of (1.3) is to be ignored for the present. The gauge-invariant curvature tensor of this theory is then defined by

$$\tfrac{1}{2}B_{ikl\cdot}^{\;\;\;\;s} = \Gamma^s_{i[l,k]} + \Gamma^t_{i[l}\Gamma^s_{k]t}, \tag{1.4}$$

together with its contractions

$$G_{ik} = B_{iks\cdot}^{\;\;\;\;s}, \qquad \tfrac{1}{2}nF_{kl} = B_{skl\cdot}^{\;\;\;\;s} = nk_{[k,l]}. \tag{1.5}$$

One also has incidentally

$$G_{[ik]} = -\tfrac{1}{4}nF_{ik}. \tag{1.6}$$

Field equations may be derived from action integrals involving gauge-invariant scalar densities $\mathfrak{L}$ as Lagrangians. In four dimensions, for instance, one might take $\mathfrak{L} = G\mathfrak{G}$, or $\mathfrak{L} = G_{ik}\,\mathfrak{G}^{ik}$, etc., where $G = g^{ik}G_{ik}$ and the densities are the corresponding scalars multiplied by

$$w = (-\det g_{ik})^{\frac{1}{2}}.$$

$\mathfrak{L}$ may be so selected that the resulting equations will be satisfied by any solution of the simple covariant set of equations

$$G_{(ik)} = 0. \tag{1.7}$$

† At the time of writing the paper referred to I was under the impression that this kind of derivative had not previously been defined. However, it appeared already in a paper by Newman in 1927 (5). Incidentally, it should be noted that the present k_i differs from that of (4) by a factor 2.

However, the spherically symmetric solutions of (1.7) in which

$$k_1 = k_2 = k_3 = 0$$

and k_4 is supposed to represent the electrostatic potential do not appear to be physically sensible [cf. (**6**)].

(*c*) Einstein's theory involves as basic field-quantity a covariant asymmetrical tensor g_{ik}. Apart from the symmetrical tensor $g_{\underline{ik}}$ it contains therefore a bivector, viz. $g_{\underset{\vee}{ik}}$, as required. The n-dimensional space H_n in which this theory operates is provided with an asymmetrical linear connexion Γ^s_{ik} which is related to the g_{ab} and their first derivatives through the set of equations†

$$g_{ik,l} - \Gamma^s_{il}\, g_{sk} - \Gamma^s_{lk}\, g_{is} = 0. \tag{1.8}$$

The equation (1.8) clearly exhibits the property of transposition-invariance, or *hermiticity*; i.e. if the transposed of g_{ab} and Γ^r_{st} be understood to be g_{ba} and Γ^r_{ts}, then (1.8) is invariant with respect to the process of simultaneously transposing all the g's and Γ's (and interchanging the free indices i and k). In the same way the requirement of hermiticity is imposed upon the other fundamental equations of this theory.

The definition of the covariant derivative of a tensor or tensor-density $\mathfrak{T}$ (indices suppressed) is formally similar to that which appears in Riemannian geometry if due attention is paid to the relative positions of the lower indices of the components of the linear connexion. Accordingly every index of $\mathfrak{T}$ is distinguished by its character during covariant differentiation. An index is called *positive*, *negative*, or *null* respectively if the corresponding term of the covariant derivative $\mathfrak{T}_{;k}$ involves $\Gamma^{\cdot}_{\cdot k}$, $\Gamma^{\cdot}_{k\cdot}$, or $\Gamma^{\cdot}_{\underline{\cdot k}}$; and the three alternatives may be indicated by placing the symbols $+$, $-$, or 0 below the index in question. By way of example the left-hand member of (1.8) is simply $g_{\underset{+}{i}\underset{-}{k};l}$. Note that, if c is the weight of $\mathfrak{T}$, then $\mathfrak{T}_{;k}$ contains a term $-c\Gamma^s_{\underline{sk}}\mathfrak{T}$, i.e. the symmetrical part only of Γ^r_{st} occurs in it.

The curvature tensor is defined as

$$\tfrac{1}{2}B_{ikl\cdot}^{s} = \Gamma^s_{i[l,k]} + \Gamma^s_{t[k}\Gamma^t_{|i|l]}. \tag{1.9}$$

† This set is not usually *postulated* but rather derived (along with the rest of the field equations) from a suitably selected action integral, the Γ^s_{ik} and g_{ab} being varied *independently* (Palatini's method). The choice of a Lagrangian appropriate to this method is however in general not possible in the gauge-invariant theory of §§ 2 ff. [cf. § 4 *b*].

Other basic tensors of importance are

$$L_{ik} = \tfrac{1}{2}B_{sik\cdot}{}^{s} = \Gamma_{[i,k]}, \qquad (\Gamma_i = \Gamma^{s}_{\underset{\vee}{is}}), \tag{1.10}$$

$$`G_{ik} = \tfrac{1}{2}(B_{iks\cdot}{}^{s}+\tilde{B}_{kis\cdot}{}^{s})-L_{ik} = -\Gamma^{s}_{ik,s}+\tfrac{1}{2}(\Gamma^{s}_{\underline{is},k}+\Gamma^{s}_{\underline{sk},i})+\Gamma^{s}_{it}\,\Gamma^{t}_{sk}-\Gamma^{s}_{ik}\,\Gamma^{t}_{\underline{st}}, \tag{1.11}$$

$$`G = g^{ik}\,`G_{ik}. \tag{1.12}$$

(The tilde indicates transposition.) It should be noted that L_{ik} and $`G_{ik}$ are both hermitian. Finally, field equations may be derived by choosing a suitable action integral, e.g. with the Lagrangian $`\mathfrak{G}$.

(*d*) The main purpose of this paper is to consider a space J_n which is provided with an asymmetrical covariant tensor g_{ik} and a covariant vector k_i; and further an asymmetrical linear connexion Γ^{s}_{ab} related to these in such a way that a J_n may be regarded equivalently as the gauge-invariant generalization of an H_n, or the 'asymmetrical' generalization of a W_n.

2. Let the set of linear algebraic equations

$$\Delta^{s}_{il}\,g_{sk}+\Delta^{s}_{lk}\,g_{is} = f_{ik} \tag{2.1}$$

have the solution (supposed to exist)

$$\Delta^{s}_{ik} = p^{sabc}_{ik}\,f_{abc}, \tag{2.2}$$

where the p^{sabc}_{ik} are evidently functions of the g_{jl} alone. Let a linear connexion Γ^{s}_{ik} be defined as

$$\Gamma^{s}_{ik} = p^{sabc}_{ik}(g_{ab,c}-g_{ab}\,k_c) = \underset{0}{\Gamma}{}^{s}_{ik}-\gamma^{s}_{ik}, \text{ say,} \tag{2.3}$$

so that $\underset{0}{\Gamma}{}^{s}_{ik}$ is the part of Γ^{s}_{ik} which does not involve the k_c. It follows at once that

$$\Gamma^{s}_{il}\,g_{sk}+\Gamma^{s}_{lk}\,g_{is} = g_{ik,l}-g_{ik}\,k_l. \tag{2.4}$$

Consider now the gauge transformation

$$g_{ab} \to \lambda g_{ab}, \qquad k_c \to k_c+(\log\lambda)_{,c}, \tag{2.5}$$

where λ is an arbitrary function of the coordinates. It follows at once from (2.1), (2.2) that in the gauge transformation p^{sabc}_{ik} multiplies itself by λ^{-1}, and therefore from (2.3) that Γ^{s}_{ik} is gauge-invariant.

Amongst the tensors which can be constructed from g_{ab}, k_c and their derivatives, only those will henceforth be considered which exhibit the property of gauge-covariance as defined in § 1 *b*; and the terms coordinate-weight and gauge-weight will have the meanings described there. At the same time fundamental equations in J_n shall be hermitian in the sense defined in § 1 *c*. The gauge-invariant covariant derivative of a tensor-density $\mathfrak{T}^{i_1\ldots i_\mu}_{k_1\ldots k_\nu}$ is now defined by (1.3) with the understanding that all

the indices of $\mathfrak{T}$ have been taken as positive, in the sense of § 1 *c*. In the case of a negative or null index the $\Gamma_{\cdot i}$ which appears in the corresponding term of the covariant derivative must of course be replaced by $\Gamma_{i\cdot}^{\cdot}$, $\Gamma_{\underline{\cdot i}}$ respectively. Because of (2.4), (1.3) one now has the important gauge-invariant hermitian equation

$$g_{\underset{+}{i}\underset{-}{k};l} = 0, \qquad (2.6)$$

g_{ik} itself being of gauge-weight $v = 1$.

It should be noted that one cannot identify the vector k_i with the vector Γ_i *a priori* since otherwise the hermiticity of the equation (2.6) will be lost.

3. Keeping in mind that w has the gauge-weight $v = \frac{1}{2}n$, we see in the usual way that

$$g^{\underset{+}{i}\underset{-}{k}}{}_{;l} = 0, \qquad \mathfrak{g}^{\underset{+}{i}\underset{-}{k}}{}_{;l} = 0. \qquad (3.1)$$

From (3.1) one gets

$$\mathfrak{g}^{\underset{\vee}{it}}{}_{,t} = \mathfrak{g}^{\underline{it}}\Gamma_t + (\tfrac{1}{2}n-1)\mathfrak{g}^{\underset{\vee}{it}}k_t, \qquad (3.2)$$

and

$$\mathfrak{g}^{\underline{it}}{}_{,t} = \mathfrak{g}^{\underset{\vee}{it}}\Gamma_t + (\tfrac{1}{2}n-1)\mathfrak{g}^{\underline{it}}k_t - \mathfrak{g}^{st}\Gamma^i_{st}. \qquad (3.3)$$

Note that $\Gamma_i = 0$ now naturally does not in general imply $\mathfrak{g}^{\underset{\vee}{is}}{}_{,s} = 0$: the latter equation is not gauge-invariant. The identity $w_{;i} = 0$ gives the useful relation

$$\Gamma^s_{\underline{is}} = (\log w)_{,i} - \tfrac{1}{2}nk_i. \qquad (3.4)$$

4. (*a*) The curvature tensor $B_{ikl\cdot}^{\ \ \ \ s}$ is defined by (1.9), the Γ^c_{ab} now being of course those of § 2. The identity of Ricci—as regards a vector T_i of gauge-weight v—then reads

$$T_{\underset{+}{i}\underset{+}{k};l} - T_{\underset{+}{i}\underset{-}{l};k} = B_{ikl\cdot}^{\ \ \ \ s}T_s - vT_iF_{kl}, \qquad (4.1)$$

where $T_{ik} = T_{\underset{+}{i};k}$ and $F_{kl} = 2k_{[k,l]}$. From the present Γ^c_{ab} one can form the *gauge-invariant hermitian Einstein tensor* [cf. (1.11)]

$$`G_{ik} = -\Gamma^s_{ik,s} + \tfrac{1}{2}(\Gamma^s_{\underline{is},k} + \Gamma^s_{\underline{sk},i}) + \Gamma^s_{it}\Gamma^t_{sk} - \Gamma^s_{ik}\Gamma^t_{\underline{st}}; \qquad (4.2)$$

and $`G$ again stands for the scalar invariant $g^{ik}\ `G_{ik}$. Notice that, when the skew-symmetrical part of g_{ik} is put equal to zero, $`G_{ik}$ reduces to the symmetrical part $G_{(ik)}$ of Weyl's tensor [cf. (1.7)].

(*b*) As in the original theories of Einstein and of Weyl one will normally wish to derive field equations from a variational principle $\delta \int \mathfrak{L}\, d\tau = 0$. The Lagrangian $\mathfrak{L}$ one naturally requires to be a gauge-invariant hermitian scalar-density: except when $n = 2$, the choice $\mathfrak{L} = `\mathfrak{G}$ is therefore excluded. On the other hand, for any $n$, $\mathfrak{L} = w\,`G^{\frac{1}{2}n}$ is admissible, as is for instance $\mathfrak{L} = (\det `G_{ik})^{\frac{1}{2}}$.

It is evident that Palatini's device can no longer be usefully applied; for, when $\mathfrak{L}$ does not depend linearly on the $`G_{ik}$, the vanishing of the variation of $\int \mathfrak{L}\, d\tau$ in an independent variation of the Γ^c_{ab} alone will not yield simple relations between the Γ^c_{ab} and the $g_{ik,l}$ akin to (2.6). For this reason the relation (2.6) was here postulated from the outset; and in the absence of subsidiary conditions the field equations may be taken to be

$$P_{ik} = 0, \qquad Q^i = 0, \tag{4.3}$$

where P_{ik}, $\mathfrak{Q}^i$ are the hamiltonian derivatives of $\mathfrak{L}$ with respect to $\mathfrak{g}^{ik}$, k_i respectively. This means that, if all variations vanish on the boundary of the region of integration, one has identically

$$\delta \int \mathfrak{L}\, d\tau = \int (P_{ik}\, \delta\mathfrak{g}^{ik} + \mathfrak{Q}^i \delta k_i)\, d\tau. \tag{4.4}$$

One may wish to adjoin the prior condition $\Gamma_i = 0$. In that case the field equations will be

$$P_{\underline{ik}} = 0, \qquad P_{[ik;l]} = 0, \qquad Q^i = 0, \qquad \Gamma_i = 0. \tag{4.5}$$

It should be noted that, for $n > 2$, the condition $\Gamma_i = 0$ implies the restriction

$$g^{ik}_{\vee} F_{ik} = 0. \tag{4.6}$$

(*c*) If one supposes the variations to arise solely from an infinitesimal gauge transformation, i.e.

$$\delta\mathfrak{g}^{ik} = (\tfrac{1}{2}n-1)\mathfrak{g}^{ik}\lambda, \qquad \delta k_i = \lambda_{,i},$$

then from the gauge-invariance of $\mathfrak{L}$ the identity

$$\mathfrak{Q}^i{}_{;i} = (\tfrac{1}{2}n-1)\mathfrak{g}^{ik}P_{ik} \tag{4.7}$$

follows easily. On the other hand one may generate $\delta\mathfrak{g}^{ik}$ and δk_i by means of an infinitesimal coordinate transformation alone. In that case, using (4.7), one obtains the identity

$$\mathfrak{g}^{st}(P_{\underset{+}{s}\,\underset{+}{i};t} + P_{\underset{-}{i}\,\underset{-}{t};s} - P_{\underset{+}{s}\,\underset{-}{t};i} + P_{si}\Gamma_t - P_{it}\Gamma_s) = F_{is}\,\mathfrak{Q}^s. \tag{4.8}$$

5. (*a*) It is desirable to demonstrate by an example how gauge-covariant hermitian hamiltonian derivatives can be obtained explicitly. For this purpose I naturally choose $n = 4$. In that case the Lagrangian $`\mathfrak{L} = w\,`G^2$ will be admissible. It turns out, however, that the Lagrangian

$$\mathfrak{L} = wG^2, \quad \text{where } G = g^{ik}G_{ik} = g^{ik}(`G_{ik} - \Gamma_{[i,k]}), \tag{5.1}$$

results in hamiltonian derivatives whose superficial appearance is rather simpler than that of the derivatives arising from $`\mathfrak{L}$. Since for the present purpose $\mathfrak{L}$ is just as appropriate as $`\mathfrak{L}$, I shall consider the former.

(*b*) The symbol $\doteq$ will be used as follows: for any A, B, $A \doteq B$ means

'A and B differ only by an expression which constitutes an ordinary divergence'. Then, taking $\mathfrak{L} = \frac{1}{2}G\mathfrak{G}$,

$$\begin{aligned}\delta\mathfrak{L} &= G\delta\mathfrak{G} - \tfrac{1}{2}G^2\delta w \\ &= GG_{ik}\,\delta\mathfrak{g}^{ik} - \tfrac{1}{2}G^2\delta w + \\ &\qquad + G\mathfrak{g}^{ik}\delta[-\Gamma^s_{ik,s} + \tfrac{1}{2}(\Gamma^s_{si,k} + \Gamma^s_{ks,i}) + \Gamma^s_{it}\Gamma^t_{sk} - \Gamma^s_{ik}\Gamma^t_{st}],\end{aligned}$$

the expression in the square brackets being just '$G_{ik} - \Gamma_{[i,k]}$. If a divergence be rejected and thereafter ordinary derivatives of the type $(G\mathfrak{g}^{ik})_{,s}$ replaced by covariant derivatives according to (1.3), one finds that

$$\begin{aligned}\delta\mathfrak{L} &\doteq (GG_{ik} - \Gamma^s_{ik}\,G_{;s})\delta\mathfrak{g}^{ik} - \tfrac{1}{2}G^2\delta w + G_{;s}\,\delta\mathfrak{q}^s - \\ &\qquad - (\mathfrak{g}^{\underline{ik}}G_{;k} + \mathfrak{g}^{\overset{ik}{\vee}}G\Gamma_k)\delta\Gamma^s_{\underline{is}} + (\mathfrak{g}^{\overset{ik}{\vee}}G_{;k} + \mathfrak{g}^{\underline{ik}}G\Gamma_k)\delta\Gamma_i, \qquad (5.2)\end{aligned}$$

where $\mathfrak{q}^s = \mathfrak{g}^{ab}\Gamma^s_{ab}$. In view of (3.3) one has

$$\begin{aligned}G_{;s}\,\delta\mathfrak{q}^s &= G_{;s}\,\delta(-\mathfrak{g}^{\underline{st}}{}_{,t} + \mathfrak{g}^{\overset{st}{\vee}}\Gamma_t + \mathfrak{g}^{\underline{st}}\,k_t) \\ &\doteq G_{;s,t}\,\delta\mathfrak{g}^{\underline{st}} + G_{;s}\,\Gamma_t\,\delta\mathfrak{g}^{\overset{st}{\vee}} + \mathfrak{g}^{\overset{st}{\vee}}\,G_{;s}\,\delta\Gamma_t + G_{;s}\,\delta(\mathfrak{g}^{\underline{st}}k_t).\end{aligned}$$

Thus (5.2) becomes

$$\begin{aligned}\delta\mathfrak{L} &\doteq [GG_{ik} + \tfrac{1}{2}(G_{;\underset{+}{i}k} + G_{;k\underset{-}{i}}) + G_{;[i}\,\Gamma_{k]}]\delta\mathfrak{g}^{ik} - \tfrac{1}{2}G^2\delta w + \\ &\qquad + \mathfrak{g}^{\underline{ik}}G\Gamma_k\,\delta\Gamma_i - (\mathfrak{g}^{\underline{ik}}G_{;k} + \mathfrak{g}^{\overset{ik}{\vee}}G\Gamma_k)\delta\Gamma^s_{\underline{is}} + \mathfrak{g}^{si}G_{;s}\,\delta k_i. \qquad (5.3)\end{aligned}$$

Now

$$\begin{aligned}\mathfrak{g}^{\underline{ik}}G\Gamma_k\,\delta\Gamma_i &= G\Gamma_k[\delta(\mathfrak{g}^{\underline{ik}}\Gamma_i) - \Gamma_i\,\delta\mathfrak{g}^{\underline{ik}}] \\ &\doteq -(G\Gamma_{[i,k]} + \Gamma_{[i}\,G_{;k]} + G\Gamma_i\,\Gamma_k)\delta\mathfrak{g}^{ik} + G\Gamma_k\,\mathfrak{g}^{\overset{ik}{\vee}}\,\delta k_i, \qquad (5.4)\end{aligned}$$

where (3.2) has been used. In view of (3.4),

$$\begin{aligned}(\mathfrak{g}^{\underline{ik}}G_{;k} + \mathfrak{g}^{\overset{ik}{\vee}}G\Gamma_k)\delta\Gamma^s_{\underline{is}} &\doteq -(\mathfrak{g}^{\underline{ik}}G_{;k} + \mathfrak{g}^{\overset{ik}{\vee}}G\Gamma_k)_{,i}\,\delta\log w - 2(\mathfrak{g}^{\underline{ik}}G_{;k} + \mathfrak{g}^{\overset{ik}{\vee}}G\Gamma_k)\delta k_i \\ &= -g^{st}[\tfrac{1}{2}(G_{;\underset{+}{s}t} + G_{;t\underset{-}{s}}) + 2G_{;[s}\,\Gamma_{t]} - G\Gamma_s\,\Gamma_t - G\Gamma_{[s,t]}]\delta w - \\ &\qquad - 2(\mathfrak{g}^{\underline{is}}G_{;s} + \mathfrak{g}^{\overset{is}{\vee}}G\Gamma_s)\delta k_i. \qquad (5.5)\end{aligned}$$

We may now insert (5.4) and (5.5) in (5.3). Keeping (4.4) in mind we can put the desired result into the form

$$\begin{aligned}P_{ik} - \tfrac{1}{6}g_{ik}P &= \tfrac{1}{2}(G_{;\underset{+}{i}k} + G_{;k\underset{-}{i}}) + 2G_{;[i}\,\Gamma_{k]} - G(\Gamma_{[i,k]} + \Gamma_i\,\Gamma_k) + G(G_{ik} - \tfrac{1}{4}g_{ik}\,G), \\ \tfrac{1}{3}Q^i &= g^{\underline{is}}G_{;s} + g^{\overset{is}{\vee}}G\Gamma_s, \qquad (5.6)\end{aligned}$$

with $P = g^{ik}P_{ik}$. It can be seen by inspection that these hamiltonian derivatives possess the required properties of coordinate-covariance, gauge-covariance, and hermiticity.

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THE LAPLACE–STIELTJES TRANSFORM OF AN INCREASING VECTOR-VALUED FUNCTION

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1. THE classical theory of the Laplace–Stieltjes transform of a numerically valued function has been extended to vector-valued functions by Hille [see, for example, Chapter X of (**4**)]. As in the classical theory, it is of interest to know in what cases the function defined by the Laplace–Stieltjes transform of a vector-valued function has a singularity on the axis of convergence of the integral. The problem was studied by Hamburger (**3**) in the case of a numerically valued function $\alpha(t)$ of the real variable t. Hamburger proved that, if $\alpha(t)$ is monotonic, then the abscissa of convergence of the integral

$$\int_0^\infty e^{-\lambda t}\,d\alpha(t) = p(\lambda)$$

is a singularity of $p(\lambda)$.

In what follows, I prove an analogue of Hamburger's theorem for an increasing function $T(t)$ of the real variable t taking its values in a partially ordered Banach space V with a closed normal cone V^+. The proof follows the same general lines as Hamburger's proof but depends to a large extent on the order properties of V.

2. Definitions

1. A function $T(t)$ defined on a set Σ and taking its values in a vector space V is called a *vector-valued function*. If Σ is a partially ordered set and V a partially ordered† vector space, we say that $T(t)$ is an *increasing function* of t if $T(t_1) \geqslant T(t_2)$ whenever $t_1 \geqslant t_2$, $t_1, t_2 \in \Sigma$.

2. Let $T(t)$ be a function defined on the closed interval $[t_1, t_2]$ of the real numbers and taking its values in a normed vector space V. Let Δ denote the set of all subdivisions of $[t_1, t_2]$ into a finite number of non-overlapping, consecutive (but not necessarily contiguous) subintervals (a_i, b_i) and Δ' the set of all possible partitions of $[t_1, t_2]$ by a finite set of points $s_1, s_2, \ldots, s_n$ with $t_1 = s_0 < s_1 < s_2 < \ldots < s_n = t_2$.

† For the definitions of partially ordered vector space, normal cone, etc., see (**1**).

We say that $T(t)$ is of

(i) *bounded variation* if

$$\sup_{\Delta}\left\|\sum_i [T(b_i)-T(a_i)]\right\| < \infty;$$

(ii) *strongly bounded variation* if

$$\sup_{\Delta'}\sum_i \|T(s_i)-T(s_{i-1})\| < \infty.$$

LEMMA 2.1. *Let V be a partially ordered normed vector space with V^+ a normal cone and constant of normality η. If $u, v \in V$ and $-u \leqslant v \leqslant u$, then*

$$\|v\| \leqslant 2\eta^{-1}\|u\|.$$

For a proof, see Lemma 3 of (**1**).

THEOREM 2.2. *Let $T(t)$ be an increasing function defined in the interval $[t_1, t_2]$ and taking its values in the partially ordered normed vector space V with V^+ a normal cone. Then $T(t)$ is of bounded variation in $[t_1, t_2]$.*

Proof. Let (a_i, b_i) $(i = 0, 1, 2,..., n)$ be a finite set of non-overlapping subintervals of $[t_1, t_2]$ and let π denote this class of subintervals of $[t_1, t_2]$. Since $T(t)$ is an increasing function, $T(b_i)-T(a_i) \in V^+$ for each i, and hence

$$\sum_{i=0}^{n}[T(b_i)-T(a_i)] \in V^+.$$

For the same reason, $\quad T(a_{i+1})-T(b_i) \in V^+,$

and therefore $\quad \sum_{i=0}^{n-1}[T(a_{i+1})-T(b_i)] \in V^+.$

Write $\quad u = \sum_{i=0}^{n}[T(b_i)-T(a_i)], \qquad v = \sum_{i=0}^{n-1}[T(a_{i+1})-T(b_i)].$

Since V^+ is a normal cone, there exists η such that

$$\|u+v\| \geqslant \eta\|u\|.$$

Now $\quad u+v = T(b_n)-T(a_0) \leqslant T(t_2)-T(t_1).$

Thus $\quad -[T(t_2)-T(t_1)] \leqslant u+v \leqslant T(t_2)-T(t_1),$

and, by Lemma 2.1,

$$\|u+v\| \leqslant 2\eta^{-1}\|T(t_2)-T(t_1)\|.$$

Hence we have

$$\|u\| \leqslant \eta^{-1}\|u+v\| \leqslant 2\eta^{-2}\|T(t_2)-T(t_1)\| = M, \text{ a constant.}$$

Since M is independent of the particular subdivision π, it follows that

$$\sup_{\Delta}\|u\| = \sup\left\|\sum_{i=0}^{n}[T(b_i)-T(a_i)]\right\| < \infty,$$

where Δ denotes the set of subdivisions of $[t_1, t_2]$ by finite sets of non-overlapping subintervals. This proves the theorem.

The following result is due to Dunford [see (**2**) 312, Theorem 11].

THEOREM 2.3. *Let f be a function of bounded variation in the interval $[a, b]$ to the Banach space X. Then the Riemann–Stieltjes integral*

$$\int_a^b \phi(s)\, df(s)$$

exists for every real-or-complex-valued continuous function ϕ.

Suppose now that V is a partially ordered Banach space. Let $T(t)$ be an increasing function defined on $[t_1, t_2]$ and taking its values in V, and let $\tilde{V}$ be the complexification [see (**1**)] of V. Then in view of Theorems 2.2 and 2.3, we obtain the following theorem:

THEOREM 2.4. *Let $f(t)$ be a continuous complex-valued function on $[t_1, t_2]$ and $T(t)$ an increasing function on $[t_1, t_2]$ to the partially ordered Banach space V with V^+ a normal cone. Then the Stieltjes integral $\int_{t_1}^{t_2} f(t)\, dT(t)$ exists as the unique limit of Riemann sums in the norm topology.*

Note. The integral exists in $\tilde{V}$.

COROLLARY. *If $\phi \in \tilde{V}^*$, then*

$$\phi\left\{\int_{t_1}^{t_2} f(t)\, dT(t)\right\} = \int_{t_1}^{t_2} f(t)\, d\phi[T(t)].$$

For the proof of the corollary, see (**4**), 52, Theorem 3.8.2.

LEMMA 2.5. *Let V be a partially ordered Banach space with V^+ a normal cone and constant of normality η. Let $T(t)$ be an increasing function defined on $[a, b]$ and taking its values in V. Suppose that the real-valued functions $f_n(t)$ $(n = 0, 1, 2, \ldots)$ are continuous in the interval $[a, b]$ and that $f_n(t) \to f(t)$ uniformly in the interval. Then*

$$\lim_{n\to\infty} \int_a^b f_n(t)\, dT(t) = \int_a^b f(t)\, dT(t).$$

COROLLARY. *If the series $\sum_{n=0}^{\infty} f_n(t)$ converges uniformly to $f(t)$ in $[a, b]$, then*

$$\int_a^b f(t)\, dT(t) = \sum_{n=0}^{\infty} \int_a^b f_n(t)\, dT(t).$$

Proof. By Theorem 2.4, the integrals

$$\int_a^b f(t)\,dT(t), \qquad \int_a^b f_n(t)\,dT(t) \quad (n = 0, 1, 2,...)$$

exist since each $f_n(t)$ is continuous and the limit function is continuous. Let K be an arbitrary partition of the interval $[a,b]$ by the points $t_0, t_1, t_2,..., t_m$, where

$$a = t_0 < t_1 < t_2 < ... < t_m = b.$$

Let $\delta = \max_{0\leqslant i\leqslant m-1} (t_{i+1}-t_i), \qquad M = \sup_{a\leqslant t\leqslant b} |f_n(t)-f(t)|.$

Then M is a non-negative real number. Consider the approximating Riemann sums

$$\sum_{i=0}^{m-1}[f_n(t_i)-f(t)][T(t_{i+1})-T(t_i)]$$

of the integral

$$\int_a^b [f_n(t)-f(t)]\,dT(t).$$

$T(t_{i+1})-T(t_i) \in V^+$ for $i = 0, 1,..., m-1$, since $T(t)$ is an increasing function. Hence

$$M[T(t_{i+1})-T(t_i)] \in V^+$$

for each i. Also $f_n(t_i)-f(t_i) \leqslant M$ for each t_i in $[a,b]$. Hence

$$|f_n(t_i)-f(t_i)|[T(t_{i+1})-T(t_i)] \leqslant M[T(t_{i+1})-T(t_i)].$$

We therefore have

$$\sum_{i=0}^{m-1}|f_n(t_i)-f(t)|[T(t_{i+1})-T(t_i)] \leqslant M \sum_{i=0}^{m-1}[T(t_{i+1})-T(t_i)]$$
$$= M[T(b)-T(a)],$$

which implies that

$$-M[T(b)-T(a)] \leqslant \sum_{i=0}^{m-1} |f_n(t_i)-f(t_i)|[T(t_{i+1})-T(t_i)] \leqslant M[T(b)-T(a)].$$

By Lemma 2.1, we have

$$\left\|\sum_{i=0}^{m-1} |f_n(t_i)-f(t_i)|[T(t_{i+1})-T(t_i)]\right\| \leqslant 2\eta^{-1} M\|T(b)-T(a)\|.$$

Since the last inequality holds for all partitions, we have

$$\left\|\int_a^b |f_n(t)-f(t)|\,dT(t)\right\| \leqslant 2\eta^{-1} M\|T(b)-T(a)\|.$$

By hypothesis, $M \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \int_a^b f_n(t)\,dT(t) = \int_a^b f(t)\,dT(t).$$

This completes the proof of the lemma and the corollary is immediate.

LEMMA 2.6. *For each $n = 0, 1, 2, \ldots$, let $a_n(R)$ be an increasing function of R on the interval $[0, \infty]$ to the partially ordered Banach space V with V^+ a normal cone which is also a closed set in V. If $a_n(R) \to a_n(\infty)$ as $R \to \infty$ for fixed n, and if*

$$\sum_{n=0}^{\infty} a_n(R) = S(R), \qquad \sum_{n=0}^{\infty} a_n(\infty) = S(\infty),$$

then $S(R) \to S(\infty)$ as $R \to \infty$.

Proof. Denote by $\{S_n(\infty)\}$ the sequence of partial sums of the $a_n(\infty)$ and by $\{S_n(R)\}$ the sequence of partial sums of the $a_n(R)$. Given $\epsilon > 0$, there exists n_0 such that $\|S(\infty) - S_n(\infty)\| < \epsilon k$ for all $n \geqslant n_0$, k being a constant positive number. With n_0 so chosen and the same ϵ, we can find R_0 such that, for all $R > R_0$,

$$\|S_{n_0}(R) - S_{n_0}(\infty)\| < \epsilon k.$$

Now, for $n \geqslant n_0$,

$$\{S_n(\infty) - S_{n_0}(\infty)\} - \{S_n(R) - S_{n_0}(R)\} \in V^+$$

since V^+ is closed. Also, as $n \to \infty$,

$$\{S_n(\infty) - S_{n_0}(\infty)\} - \{S_n(R) - S_{n_0}(R)\}$$
$$\to \{S(\infty) - S_{n_0}(\infty)\} - \{S(R) - S_{n_0}(R)\} \in V^+$$

for the same reason. Thus we have

$$0 \leqslant S(R) - S_{n_0}(R) \leqslant S(\infty) - S_{n_0}(\infty)$$

and so

$$-[S(\infty) - S_{n_0}(\infty)] \leqslant S(R) - S_{n_0}(R) \leqslant S(\infty) - S_{n_0}(\infty).$$

Hence, by Lemma 2.1,

$$\|S(R) - S_{n_0}(R)\| \leqslant 2\eta^{-1} \|S(\infty) - S_{n_0}(\infty)\| < 2\eta^{-1} \epsilon k,$$

where η is the constant of normality. Thus

$$\begin{aligned} \|S(R) - S(\infty)\| &= \|S_{n_0}(\infty) - S(\infty) + S_{n_0}(R) - S_{n_0}(\infty) + S(R) - S_{n_0}(R)\| \\ &\leqslant \|S_{n_0}(\infty) - S(\infty)\| + \|S_{n_0}(R) - S_{n_0}(\infty)\| + \|S(R) - S_{n_0}(R)\| \\ &< \epsilon k + \epsilon k + 2\eta^{-1} \epsilon k \\ &= \epsilon k(2 + 2\eta^{-1}). \end{aligned}$$

Taking $k = 1/[2 + 2\eta^{-1}]$, we have $\|S(R) - S(\infty)\| < \epsilon$ if $R > R_0$, and the lemma is proved.

3. The Laplace–Stieltjes transform

Let V be a complex Banach space and $T(t)$ a function of bounded variation in every finite closed subinterval of $[0,\infty)$ and taking its values in V. Consider the integral

$$T(t;\lambda) = \int_0^t e^{-\lambda s}\, dT(s),$$

where λ is a finite complex number and t is finite and positive. By Theorem 2.3, this integral exists. If for a given λ, $\lim_{t\to\infty} T(t;\lambda)$ exists as an element of V, we denote the limit by

$$L \equiv f(\lambda) = \int_0^\infty e^{-\lambda t}\, dT(t).$$

The integral L is then said to 'converge' for this value of λ and is called the *Laplace–Stieltjes transform* of $T(t)$.

The following theorem is proved in (**5**).

THEOREM 3.1. *Let $T(t)$ be an increasing function on $[0,\infty)$ to the partially ordered Banach space V with V^+ a normal cone. Then there exists ξ_0 $(-\infty \leqslant \xi_0 \leqslant \infty)$ such that the integral L is convergent for all λ with $\operatorname{re}\lambda > \xi_0$ but not for any λ with $\operatorname{re}\lambda < \xi_0$, and*

$$\xi_0 = \limsup_{t\to\infty} t^{-1} \log\|T(t)-T(\infty)\|.$$

The number ξ_0 is known as the *abscissa of convergence* of the integral L.

This result should be compared with Theorem 10.2.1 of Hille (**4**), where $T(t)$ is a function of strongly bounded variation and V an arbitrary complex Banach space.

4. For the proof of the main result, we need the following lemmas:

LEMMA 4.1. *Let $F(z)$ be a function on the domain D to the complex Banach space X. If $F(z)$ is regular† in D, then $F(z)$ has strong derivatives of all orders and the Taylor expansion*

$$\sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(z')(z-z')^n$$

converges uniformly to $F(z)$ for z in any circle $|z-z'| \leqslant r$ inside D.

This is Theorem 76 of Dunford (**2**).

† For the definitions of regularity, strong derivatives, etc., of a vector-valued function, see Hille (**4**) chapter 3.

LEMMA 4.2. *Let $\alpha(t)$ be a complex-valued function of bounded variation in every finite subinterval of the interval $0 \leqslant t < \infty$ and let*

$$\xi_0 = \limsup_{t\to\infty} t^{-1}\log\|\alpha(t)-\alpha(\infty)\|.$$

If the integral

$$p(\lambda) = \int_0^\infty e^{-\lambda t}\,d\alpha(t)$$

converges for all λ with $\operatorname{re}\lambda > \xi_0$ *($\xi_0 < \infty$), then $p(\lambda)$ is regular for all λ with* $\operatorname{re}\lambda > \xi_0$ *and*

$$p^{(k)}(\lambda) = \int_0^\infty e^{-\lambda t}(-t)^k\,d\alpha(t).$$

For a proof, see (**6**) 57.

LEMMA 4.3. *Let $T(t)$ be as defined in Theorem* 3.1 *and ξ_0 the abscissa of convergence of the integral $\int_0^\infty e^{-\lambda t}\,dT(t)$. Then the function*

$$f(\lambda) = \int_0^\infty e^{-\lambda t}\,dT(t)$$

is a regular function of λ in the half-plane $\operatorname{re}\lambda > \xi_0$ *and the derivatives of $f(\lambda)$ are given by*

$$f^{(n)}(\lambda) = (-1)^n\int_0^\infty e^{-\lambda t}t^n\,dT(t) \quad (\operatorname{re}\lambda > \xi_0;\ n = 0, 1, 2,...).$$

Proof. First consider the integral

$$T(t;\lambda) = \int_0^t e^{-\lambda s}\,dT(s)$$

defined on the finite interval $[0,t]$. Let $\phi \in \tilde{V}^*$; then, by the corollary to Theorem 2.4,

$$\phi\left\{\int_0^t e^{-\lambda s}\,dT(s)\right\} = \int_0^t e^{-\lambda s}\,d\phi[T(s)]. \tag{1}$$

For a fixed λ with $\operatorname{re}\lambda > \xi_0$, $T(t;\lambda)$ tends to the limit $f(\lambda)$ in $\tilde{V}$ as $t \to \infty$, and, by the continuity of ϕ,

$$\phi[T(t;\lambda)] \to \phi[f(\lambda)] \quad (t\to\infty).$$

But, as $t \to \infty$, the limit of the right-hand side of (1) is

$$\int_0^\infty e^{-\lambda t}\,d\phi[T(t)].$$

Hence

$$\phi[f(\lambda)] = \int_0^\infty e^{-\lambda t}\,d\phi[T(t)] \quad (\operatorname{re}\lambda > \xi_0).$$

Let ξ_ϕ be the abscissa of convergence of the integral

$$\int_0^\infty e^{-\lambda t}\, d\phi[T(t)].$$

Then

$$\begin{aligned}\xi_\phi &= \limsup_{t\to\infty} t^{-1}\log\|\phi[T(t)]-\phi[T(\infty)]\| \\ &\leqslant \limsup_{t\to\infty} t^{-1}\log\{\|\phi\|\,\|T(t)-T(\infty)\|\} \\ &= \xi_0.\end{aligned}$$

We now apply Lemma 4.2 to the numerically valued function

$$g(\lambda) = \phi[f(\lambda)] = \int_0^\infty e^{-\lambda t}\, d\phi[T(t)].$$

Then $g(\lambda)$ is regular for all λ with $\operatorname{re}\lambda > \xi_\phi$ and hence for all λ with $\operatorname{re}\lambda > \xi_0$, and we have

$$g^{(n)}(\lambda) = \frac{d^n}{d\lambda^n}\phi[f(\lambda)] = \int_0^\infty e^{-\lambda t}(-t)^n\, d\phi[T(t)]$$

for all λ with $\operatorname{re}\lambda > \xi_0$ and $n = 0, 1, 2,\ldots$. Also, since

$$\phi[f^{(n)}(\lambda)] = \frac{d^n}{d\lambda^n}\phi[f(\lambda)] = g^{(n)}(\lambda),$$

we have, for each n,

$$\phi[f^{(n)}(\lambda)] = \int_0^\infty e^{-\lambda t}(-t)^n\, d\phi[T(t)] = \phi\left\{\int_0^\infty e^{-\lambda t}(-t)^n\, dT(t)\right\}.$$

This being true for all $\phi \in \tilde{V}^*$, we obtain

$$f^{(n)}(\lambda) = (-1)^n \int_0^\infty e^{-\lambda t}t^n\, dT(t) \quad (\operatorname{re}\lambda > \xi_0;\ n = 0, 1, 2,\ldots).$$

The following is the main theorem:

THEOREM 4.4. *Let V be a partially ordered Banach space with V^+ a closed normal cone, and $T(t)$ an increasing function defined on $[0,\infty)$ and taking its values in V. Then the abscissa of convergence ξ_0 of the Laplace-Stieltjes transform*

$$\int_0^\infty e^{-\lambda t}\, dT(t) = f(\lambda)$$

of $T(t)$ is a singularity of $f(\lambda)$.

Proof. We first remark that, from the statement of the theorem, the only case we need consider is that in which ξ_0 is finite. Suppose that $f(\lambda)$ is regular at the point $\lambda = \xi_0$. Then, by Lemma 4.1, there is a circle

$|\lambda-\xi| < r$, with centre ξ ($> \xi_0$) and radius r ($\xi-r < \xi_0$), in which the Taylor expansion

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(\xi)(\lambda-\xi)^n$$

converges uniformly in the norm topology. In particular, the series converges for some real value of

$$\lambda = \xi-\delta < \xi_0 \quad (\delta > 0).$$

By Lemma 4.3, $$f^{(n)}(\xi) = (-1)^n \int_0^{\infty} e^{-\xi t} t^n \, dT(t).$$

We therefore have

$$f(\xi-\delta) = \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} e^{-\xi t} t^n \, dT(t) \frac{(\xi-\delta-\xi)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\infty} e^{-\xi t} t^n \, dT(t)(-\delta)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(-\delta)^n}{n!} \int_0^{\infty} e^{-\xi t} t^n \, dT(t).$$

Consider the integral $$\int_0^R e^{(\delta-\xi)t} \, dT(t).$$

We wish to show that, as $R \to \infty$,

$$\int_0^R e^{(\delta-\xi)t} \, dT(t) \to f(\xi-\delta) = \sum_{n=0}^{\infty} (-1)^n \frac{(-\delta)^n}{n!} \int_0^{\infty} e^{-\xi t} t^n \, dT(t).$$

We have $$e^{\delta t} = \sum_{n=0}^{\infty} (-1)^n \frac{\delta^n}{n!} (-t)^n,$$

and this series is dominated by

$$\sum_{n=0}^{\infty} \frac{\delta^n}{n!} R^n \quad (0 \leqslant t \leqslant R).$$

Hence, $$\sum_{n=0}^{\infty} (-1)^n \frac{\delta^n}{n!} (-t)^n$$

converges uniformly in the interval $[0, R]$, and the same is therefore true of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{\delta^n}{n!} (-t)^n e^{-\xi t}.$$

Since the functions

$$(-1)^n \frac{\delta^n}{n!}(-t)^n e^{-\xi t} \quad (n = 0, 1, 2,...)$$

are continuous real-valued functions of t on $[0, R]$ and

$$\sum_{n=0}^{\infty} (-1)^n \frac{\delta^n}{n!}(-t)^n e^{-\xi t}$$

converges uniformly to $e^{(\delta-\xi)t}$ $(0 \leqslant t \leqslant R)$, we have, by the corollary to Lemma 2.5,

$$\int_0^R e^{(\delta-\xi)t}\, dT(t) = \sum_{n=0}^{\infty} \int_0^R (-1)^n \frac{\delta^n}{n!}(-t)^n e^{-\xi t}\, dT(t)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\delta^n}{n!} \int_0^R (-t)^n e^{-\xi t}\, dT(t). \tag{2}$$

Using the notation in Lemma 2.6, we now write

$$a_n(R) = (-1)^n \frac{\delta^n}{n!} \int_0^R (-t)^n e^{-\xi t}\, dT(t),$$

$$S(R) = \sum_{n=0}^{\infty} (-1)^n \frac{\delta^n}{n!} \int_0^R (-t)^n e^{-\xi t}\, dT(t),$$

$$S(\infty) = \sum_{n=0}^{\infty} (-1)^n \frac{\delta^n}{n!} \int_0^{\infty} (-t)^n e^{-\xi t}\, dT(t).$$

It then follows from Lemma 2.6 that the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{\delta^n}{n!} \int_0^R (-t)^n e^{-\xi t}\, dT(t)$$

converges in norm to

$$\sum_{n=0}^{\infty} (-1)^n \frac{\delta^n}{n!} \int_0^{\infty} (-t)^n e^{-\xi t}\, dT(t).$$

By (2), this means that

$$\int_0^R e^{(\delta-\xi)t}\, dT(t) \to \sum_{n=0}^{\infty} (-1)^n \frac{(-\delta)^n}{n!} \int_0^{\infty} e^{-\xi t} t^n\, dT(t) = f(\xi-\delta)$$

as $R \to \infty$, and hence,

$$f(\xi-\delta) = \int_0^\infty e^{-(\xi-\delta)t}\, dT(t).$$

Thus the Laplace-integral representation of $f(\lambda)$ converges for

$$\lambda = \xi-\delta < \xi_0.$$

But this is impossible since ξ_0 is the abscissa of convergence of $f(\lambda)$. This contradiction arises from assuming that $f(\lambda)$ is regular at ξ_0; hence ξ_0 must be a singularity of $f(\lambda)$.

Example. As an example of the situation in the theorem, we take V to be the space L^p $(1 \leqslant p \leqslant \infty)$ of all pth-power summable functions defined on the real line and $V^+ = (L^p)^+$ as the set of all $f \in L^p$ with $f(s) \geqslant 0$ p.p. With the usual norm in L^p, it is not difficult to show that the partially ordered Banach space L^p has $(L^p)^+$ as a closed normal cone. Then any function $T(s,t)$ of the two variables s, t $(t \in [0,\infty))$, which belongs to L^p for each t, with the property $T(s,t') \geqslant T(s,t)$ whenever $t' \geqslant t$, satisfies the hypotheses of the theorem.

Our result then asserts in this case that the abscissa of convergence of the Laplace–Stieltjes transform $\int_0^\infty e^{-\lambda t}\, d_t T(s,t)$ is a singularity of the function $f(s,\lambda)$. By taking a fixed s, we obtain Hamburger's result for the integral

$$f(.,\lambda) = \int_0^\infty e^{-\lambda t}\, dT(.,t), \tag{3}$$

but we note that the result cannot be inferred from the consideration of the abscissa of convergence of the right-hand side of (3).

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FUNCTIONS OF BOUNDED VARIATION IN TOPOLOGICAL VECTOR SPACES

By J. D. WESTON (*Newcastle upon Tyne*)

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1. FOR functions of a real variable which take their values in a general topological vector space, the notion of total variation is not available; it is nevertheless possible to give a natural meaning to the term 'function of bounded variation'. In the case of a partially ordered topological vector space, the question then arises as to whether all monotonic functions on a finite closed interval are of bounded variation. It will be shown that this is so if the space satisfies a certain condition which relates its topological structure to its order structure (generalizing the concept of 'normality' of the positive cone in a partially ordered normed space). Locally convex spaces that satisfy this condition have recently been studied, in another context, by F. F. Bonsall (**1**).

The integral of a continuous scalar-valued function with respect to a function of bounded variation, on a finite interval, can be defined in a topological vector space which is locally convex and sequentially complete: it will be shown, in fact, that the existing definition for Banach spaces is already sufficiently general. Hence, for example, in a sequentially complete space of the type considered by Bonsall, the Laplace–Stieltjes transform of a monotonic function can be defined in a natural way; and some results concerning this which have been obtained by A. Olubummo (**2**) [see above] remain valid under this generalization.

2. Let $[a, b]$ be a real interval, and let S be a finite succession of non-overlapping closed subintervals $[a_i, b_i]$ $(i = 1,..., m)$. If $g(t)$ is a function on $[a, b]$ with values in a vector space X, let

$$v_S(g) = \sum_{i=1}^{m} \{g(b_i) - g(a_i)\},$$

and let $V(g)$ be the set of points $v_S(g)$, for all possible choices of S. If X is a topological vector space, we can say that $g(t)$ is *of bounded variation* in $[a, b]$ if $V(g)$ is a bounded set (that is, if, when G is any neighbourhood of the origin in X, there is a positive scalar ρ such that $\rho V(g) \subseteq G$). This definition agrees with that given in (**3**) 39 when the topology of X is determined by a norm.

Among the partially ordered topological vector spaces, many of the

most interesting examples satisfy a condition which can be described by saying that *there are arbitrarily small neighbourhoods of the origin which are intervals*: that is, corresponding to any neighbourhood G of the origin there is a neighbourhood H of the origin such that $x \in G$ whenever there exist u and v in H with $u \leqslant x \leqslant v$ [cf. (**1**)]. It is not difficult to see that for normed spaces this condition is equivalent to the existence of a 'constant of normality' (a positive number η such that $\|u+v\| \geqslant \eta\|u\|$ for all positive u and v). Spaces that satisfy the condition may conveniently be referred to as 'A-spaces'.

If $g(t)$ is a monotonic function on a finite real interval $[a,b]$, with values in an A-space, then $g(t)$ is of bounded variation in $[a,b]$. For, if S is a finite succession of non-overlapping closed subintervals of $[a,b]$, and S' is the succession of complementary subintervals, we have

$$v_S(g)+v_{S'}(g) = g(b)-g(a);$$

and, if $g(t)$ is non-decreasing (as we may suppose), then $v_S(g) \geqslant 0$ and $v_{S'}(g) \geqslant 0$, so that

$$0 \leqslant v_S(g) \leqslant g(b)-g(a).$$

Now let H be a neighbourhood of the origin which is an interval. A positive scalar ρ exists such that $\rho[g(b)-g(a)] \in H$; hence $\rho v_S(g) \in H$, and therefore $\rho V(g) \subseteq H$. Since H is arbitrarily small, it follows that $V(g)$ is a bounded set. [Cf. (**2**), Theorem 2.2.]

3. In a locally convex space X which is sequentially complete, it is possible to define the 'Riemann–Stieltjes' integral $\int_a^b f(t)\,dg(t)$ of a scalar-valued continuous function $f(t)$ with respect to a function $g(t)$ of bounded variation in a finite interval $[a,b]$. In fact the integral exists in a Banach space, in the manner defined by Dunford (**4**). For, let C be the closed symmetric convex hull of $V(g)$, and let L be the subspace of X generated by C. Since X is sequentially complete, the same is true of the closed set C. It follows that L can be normed in such a way that it becomes a Banach space, with C as the unit ball, the norm topology being stronger than (or possibly identical with) the induced topology [for a proof of this, see (**5**), Lemma 2]. Then $g(t)$ is a function of bounded variation with respect to the norm in L. Now, for any function $f(t)$, all 'Riemann sums'

$$\sum_{i=1}^{m} f(\tau_i)\{g(t_i)-g(t_{i-1})\},$$

where $\quad a = t_0 < t_1 < \ldots < t_n = b \quad$ and $\quad t_{i-1} \leqslant \tau_i \leqslant t_i,$

belong to L. Consequently, if $f(t)$ is continuous, the integral $\int_a^b f(t)\,dg(t)$ exists as a point of L to which the Riemann sums are arbitrarily close with respect to the norm, and hence also with respect to the topology of X.

For any two continuous functions $f_1(t)$, $f_2(t)$, and scalars α_1, α_2, we have

$$\int_a^b \{\alpha_1 f_1(t)+\alpha_2 f_2(t)\}\,dg(t) = \alpha_1 \int_a^b f_1(t)\,dg(t) + \alpha_2 \int_a^b f_2(t)\,dg(t).$$

Also, if ϕ is a continuous linear functional on X, the restriction of ϕ to L is continuous with respect to the norm, so that, by Dunford's theory,

$$\phi\left\{\int_a^b f(t)\,dg(t)\right\} = \int_a^b f(t)\,d\phi\{g(t)\},$$

where the right-hand side is an ordinary Riemann–Stieltjes integral.

In an A-space which is locally convex and sequentially complete, continuous functions can be integrated with respect to a monotonic function $g(t)$. In this case, if $\{f_n(t)\}$ is a sequence of continuous functions converging to $f(t)$ uniformly in $[a,b]$, then

$$\int_a^b f(t)\,dg(t) = \lim_{n\to\infty} \int_a^b f_n(t)\,dg(t).$$

For, let H be a neighbourhood of the origin which is an interval. Since $f(t)$ is continuous, $\int_a^b \{f_n(t)-f(t)\}\,dg(t)$ exists for each n, and, for a suitable partition of $[a,b]$,

$$\{f_n(\tau_i)-f(\tau_i)\}\{g(t_i)-g(t_{i-1})\} - \int_a^b \{f_n(t)-f(t)\}\,dg(t) \in H.$$

Now a positive number ϵ exists such that H contains the sets $\pm\epsilon V(g)$; and, if n is large enough, $|f_n(\tau_i)-f(\tau_i)| < \epsilon$ for each i. Thus, since $g(t_i)-g(t_{i-1}) \geqslant 0$ for each i,

$$\begin{aligned} -\epsilon \sum_{i=1}^{m} \{g(t_i)-g(t_{i-1})\} &\leqslant \sum_{i=1}^{m} \{f_n(\tau_i)-f(\tau_i)\}\{g(t_i)-g(t_{i-1})\} \\ &\leqslant \epsilon \sum_{i=1}^{m} \{g(t_i)-g(t_{i-1})\}, \end{aligned}$$

and the extreme terms belong to H. Hence the middle term belongs to H. Since H is arbitrarily small and n is independent of the partition,

it follows that

$$\lim_{n\to\infty}\int_a^b \{f_n(t)-f(t)\}\,dg(t) = 0.$$

[This result has been obtained for Banach spaces by Olubummo (**2**).]

4. If $g(t)$ is of bounded variation in every finite interval $[0,T]$ and has values in a sequentially complete locally convex space X, the integral $\int_0^\infty f(t)\,dg(t)$ can be defined as

$$\lim_{T\to\infty}\int_0^T f(t)\,dg(t)$$

when this limit exists in the topology of X. In particular, if $g(t)$ is monotonic in $[0,\infty)$, with values in an A-space which is locally convex and sequentially complete, the Laplace–Stieltjes transform of $g(t)$ can be defined as

$$\int_0^\infty e^{-\lambda t}\,dg(t),$$

for those values of λ at which the integral converges. If X admits complex scalars, the integral converges, if at all, in a half-plane to the right of an 'abscissa of convergence', and is a regular function of λ (in an obvious sense) in this domain. Moreover, if the positive cone in X is closed, the abscissa of convergence is a singularity of the transform. These results have been proved for a Banach space by Olubummo (**2**), whose methods can be adapted to the more general situation without difficulty.

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INVARIANCE OF THE RANK OF A PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER, UNDER CONTACT TRANSFORMATION

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CONSIDER a partial differential equation of the second order, with one dependent variable z and n independent variables $x_1,..., x_n$,

$$f(x_1,..., x_n, z, p_1,..., p_n, p_{11},..., p_{nn}) = 0,$$

where $p_i = \partial z/\partial x_i$, $p_{ij} = \partial^2 z/\partial x_i \partial x_j$. We suppose that f is an analytic function of its arguments, in the neighbourhood of a set of initial values, satisfying the equation, and that the equation is of fully reduced form, so that it may be supposed solved for one of its arguments. We define the *rank* of the equation to be the rank of the matrix

$$\begin{bmatrix} \partial f/\partial p_{11} & \frac{1}{2}\partial f/\partial p_{12} & . & . & . & \frac{1}{2}\partial f/\partial p_{1n} \\ . & . & . & . & . & . \\ \frac{1}{2}\partial f/\partial p_{n1} & . & . & . & . & \partial f/\partial p_{nn} \end{bmatrix},$$

i.e. the matrix $[c_{ik}]$, where

$$c_{ii} = \partial f/\partial p_{ii}, \qquad c_{ik} = c_{ki} = \tfrac{1}{2}\partial f/\partial p_{ik} = \tfrac{1}{2}\partial f/\partial p_{ki} \quad (i \neq k).$$

I have shown† that in the case when $n = 3$ the equation possesses one family of one-dimensional characteristics, suitably defined, if it be of rank 1; two distinct families if it be of rank 2; but no one-dimensional characteristics if it be of rank 3. It can also, by similar reasoning, be shown that, in the general case, the same result is true, i.e. if the equation be of rank 1, it has one family of one-dimensional characteristics; if it be of rank 2, it has two distinct families; but, if it be of rank 3 or more, it has no characteristics of this kind.

Now, if a contact transformation be applied, it is evident that characteristics transform into characteristics. Hence it is to be expected that an equation of rank 1 transforms into an equation of rank 1, an equation of rank 2 into an equation of rank 2, but that an equation of rank 3 or more would transform into an equation of rank 3 or more, the rank not necessarily being the same. In this paper, I shall establish a stronger

† D.Phil. thesis (Oxford, 1952).

result, namely that the rank of the equation is invariant under contact transformation, whether it be greater than 2 or not.

Suppose that we have a contact transformation relating the variables (x_i, z, p_i) and (X_i, Z, P_i) $(i = 1,..., n)$, such that

$$dZ - \sum_j P_j\, dX_j = \rho\Big(dz - \sum_j p_j\, dx_j\Big),$$

where $\rho \neq 0$. If ψ be any function of $x_1,..., x_n, z, p_1,..., p_n$, we write

$$\frac{d\psi}{dx_i} = \frac{\partial\psi}{\partial x_i} + p_i \frac{\partial\psi}{\partial z}.$$

Then it is a standard result that

$$\sum_i \left(\frac{\partial X_i}{\partial p_k}\frac{\partial P_i}{\partial p_j} - \frac{\partial X_i}{\partial p_j}\frac{\partial P_i}{\partial p_k}\right) = 0, \tag{1}$$

$$\sum_i \left(\frac{dX_i}{dx_k}\frac{dP_i}{dx_j} - \frac{dX_i}{dx_j}\frac{dP_i}{dx_k}\right) = 0, \tag{2}$$

$$\sum_i \left(\frac{\partial P_i}{\partial p_k}\frac{dX_i}{dx_j} - \frac{dP_i}{dx_j}\frac{\partial X_i}{\partial p_k}\right) = \rho\delta_{jk}, \tag{3}$$

δ_{jk} being Kronecker's delta.

To extend the transformation to elements of contact of the second order, we have identically, for any variables P_{ij},

$$\begin{aligned} dP_i - \sum_j P_{ij}\, dX_j \\ &= \sum_k \left(\frac{\partial P_i}{\partial x_k} - \sum_j P_{ij}\frac{\partial X_j}{\partial x_k}\right) dx_k + \left(\frac{\partial P_i}{\partial z} - \sum_j P_{ij}\frac{\partial X_j}{\partial z}\right) dz + \\ &\qquad + \sum_m \left(\frac{\partial P_i}{\partial p_m} - \sum_j P_{ij}\frac{\partial X_j}{\partial p_m}\right) dp_m \\ &= \left(\frac{\partial P_i}{\partial z} - \sum_j P_{ij}\frac{\partial X_j}{\partial z}\right)\Big(dz - \sum_k p_k\, dx_k\Big) + \\ &\quad + \sum_m \left(\frac{\partial P_i}{\partial p_m} - \sum_j P_{ij}\frac{\partial X_j}{\partial p_m}\right)\Big(dp_m - \sum_k p_{mk}\, dx_k\Big) + \\ &\quad + \sum_k \left\{\frac{dP_i}{dx_k} - \sum_j P_{ij}\frac{dX_j}{dx_k} + \sum_m p_{mk}\left(\frac{\partial P_i}{\partial p_m} - \sum_j P_{ij}\frac{\partial X_j}{\partial p_m}\right)\right\} dx_k. \end{aligned}$$

Define the variables P_{ij} so as to make the coefficient of each dx_k zero, i.e. by the equations

$$\sum_j \left(\frac{dX_j}{dx_k}+\sum_m p_{mk}\frac{\partial X_j}{\partial p_m}\right)P_{ij} = \frac{dP_i}{dx_k}+\sum_m p_{mk}\frac{\partial P_i}{\partial p_m} \quad (k,\, i = 1,\dots, n). \tag{4}$$

Let us now write $$a_{kj} = \frac{dX_j}{dx_k}+\sum_m p_{mk}\frac{\partial X_j}{\partial p_m}.$$

Then, in order that $X_1,\dots, X_n$ may be regarded as independent variables after the transformation, it is necessary that the latter be such that the determinant $\Delta \equiv \|a_{kj}\|$, with a_{kj} in the kth row and jth column, for general elements of contact of the second order, be non-zero. Let A_{kj} denote the cofactor of a_{kj}, divided by Δ, so that

$$\sum_k a_{kj} A_{kl} = \delta_{jl}.$$

Then, solving the equations (4), we have

$$P_{ij} = \sum_l A_{lj}\left(\frac{dP_i}{dx_l}+\sum_m p_{ml}\frac{\partial P_i}{\partial p_m}\right). \tag{5}$$

In passing, we rapidly establish that $P_{ij} = P_{ji}$ as an algebraic consequence of this definition, without regarding P_{ij} as denoting

$$\partial^2 Z/\partial X_i \partial X_j.$$

Interchanging i and j in (5), multiplying by a_{kj}, summing on j, and using the relations (1)–(3) above, we obtain

$$\sum_j a_{kj} P_{ji} = \frac{dP_i}{dx_k}+\sum_m p_{mk}\frac{\partial P_i}{\partial p_m} \quad (i,\, k = 1,\dots, n),$$

which are the same as (4), with P_{ji} written for P_{ij}. It follows that

$$P_{ij} = P_{ji}.$$

Now, regarding the variables P_{ij} as functions of the variables p_{rs}, we differentiate (4) with respect to p_{rs} $(r \neq s)$, and obtain (noticing that $p_{rs} = p_{sr}$)

$$\left.\begin{aligned} &\sum_j a_{kj}\frac{\partial P_{ij}}{\partial p_{rs}} = 0 \quad (k \neq r,\, s)\\ &\sum_j a_{rj}\frac{\partial P_{ij}}{\partial p_{rs}} = \frac{\partial P_i}{\partial p_s}-\sum_j P_{ij}\frac{\partial X_j}{\partial p_s}\\ &\sum_j a_{sj}\frac{\partial P_{ij}}{\partial p_{rs}} = \frac{\partial P_i}{\partial p_r}-\sum_j P_{ij}\frac{\partial X_j}{\partial p_r} \end{aligned}\right\}. \tag{6}$$

But, substituting for P_{ij} from (5), with i and j interchanged, and using (1)–(3), we have

$$\begin{aligned}
\frac{\partial P_i}{\partial p_s}-\sum_j P_{ij}\frac{\partial X_j}{\partial p_s} &= \frac{\partial P_i}{\partial p_s}-\sum_j\left\{\frac{\partial X_j}{\partial p_s}\sum_t A_{ti}\left(\frac{dP_j}{dx_t}+\sum_m p_{mt}\frac{\partial P_j}{\partial p_m}\right)\right\}\\
&= \frac{\partial P_i}{\partial p_s}-\sum_t A_{ti}\left\{\sum_j \frac{\partial X_j}{\partial p_s}\frac{dP_j}{dx_t}+\sum_m p_{mt}\sum_j\frac{\partial X_j}{\partial p_s}\frac{\partial P_j}{\partial p_m}\right\}\\
&= \frac{\partial P_i}{\partial p_s}-\sum_t A_{ti}\left\{\sum_j \frac{dX_j}{dx_t}\frac{\partial P_j}{\partial p_s}-\rho\delta_{st}+\sum_m p_{mt}\sum_j\frac{\partial X_j}{\partial p_m}\frac{\partial P_j}{\partial p_s}\right\}\\
&= \frac{\partial P_i}{\partial p_s}-\sum_t A_{ti}\left\{\sum_j a_{tj}\frac{\partial P_j}{\partial p_s}-\rho\delta_{st}\right\}\\
&= \frac{\partial P_i}{\partial p_s}-\sum_j \delta_{ij}\frac{\partial P_j}{\partial p_s}+\rho\sum_t A_{ti}\delta_{st}\\
&= \rho A_{si}.
\end{aligned} \tag{7}$$

Thus (6) becomes

$$\left.\begin{aligned}
\sum_j a_{kj}\frac{\partial P_{ij}}{\partial p_{rs}} &= 0 \quad (k\neq r,s)\\
\sum_j a_{rj}\frac{\partial P_{ij}}{\partial p_{rs}} &= \rho A_{si}\\
\sum_j a_{sj}\frac{\partial P_{ij}}{\partial p_{rs}} &= \rho A_{ri}
\end{aligned}\right\}. \tag{8}$$

Solving (8), we thus have

$$\frac{\partial P_{ij}}{\partial p_{rs}} = \rho(A_{si}A_{rj}+A_{ri}A_{sj}). \tag{9}$$

Again, differentiating (4) with respect to p_{rr}, and using (7), we have

$$\sum_j a_{kj}\frac{\partial P_{ij}}{\partial p_{rr}} = 0 \quad (k\neq r),$$

$$\sum_j a_{rj}\frac{\partial P_{ij}}{\partial p_{rr}} = \rho A_{ri},$$

and thus, solving, $$\frac{\partial P_{ij}}{\partial p_{rr}} = \rho A_{ri}A_{rj}. \tag{10}$$

Suppose now that we have a partial differential equation of the second order,

$$\begin{aligned}
&f(x_1,\dots,x_n,z,p_1,\dots,p_n;p_{11},\dots,p_{nn})\\
&\qquad \equiv F(X_1,\dots,X_n,Z,P_1,\dots,P_n;P_{11},\dots,P_{nn}) = 0.
\end{aligned} \tag{11}$$

Then, by the usual rules of differentiation, remembering that $P_{ij} = P_{ji}$, we have

$$\frac{\partial f}{\partial p_{rs}} = \sum_{i=1}^{n} \frac{\partial F}{\partial P_{ii}} \frac{\partial P_{ii}}{\partial p_{rs}} + \sum_{i>j}\sum \frac{\partial F}{\partial P_{ij}} \frac{\partial P_{ij}}{\partial p_{rs}}. \tag{12}$$

Let us now write

$$C_{ii} = \partial F/\partial P_{ii}, \qquad C_{ij} = C_{ji} = \tfrac{1}{2}\partial F/\partial P_{ij} \quad (i \neq j),$$
$$c_{rr} = \partial f/\partial p_{rr}, \qquad c_{rs} = c_{sr} = \tfrac{1}{2}\partial f/\partial p_{rs} \quad (r \neq s).$$

Then, rearranging the sum on the right of (12), and using (9) and (10), we have

$$c_{rr} = \rho \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ri} C_{ij} A_{rj}, \tag{13}$$

$$c_{rs} = \tfrac{1}{2}\rho \sum_{i=1}^{n} \sum_{j=1}^{n} (A_{si} A_{rj} + A_{ri} A_{sj}) C_{ij} \quad (r \neq s). \tag{14}$$

But, since $C_{ij} = C_{ji}$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{si} C_{ij} A_{rj} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ri} C_{ij} A_{sj}.$$

Thus, from (13) and (14), we have, for every r and s,

$$c_{rs} = \rho \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ri} C_{ij} A_{sj}, \tag{15}$$

and from (15) we see that we have

$$[c_{ik}] = [A_{ik}] \times [\rho C_{ik}] \times [A_{ki}]. \tag{16}$$

But $\|a_{ik}\| \neq 0$, and therefore $\|A_{ik}\| \neq 0$. Thus $[A_{ik}]$ and $[A_{ki}]$ are non-singular matrices, and hence, from (16), ρ being non-zero, we see that the rank of $[c_{ik}]$ is the same as the rank of $[C_{ik}]$. But the ranks of these two matrices are respectively the rank of the equation (11) in the variables x_i, etc., and in the variables X_i, etc. The rank is therefore unaltered by contact transformation.

RINGS OF INFINITE MATRICES

By H. S. ALLEN (*London*)

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LET α be a sequence space and let $\Sigma(\alpha)$ denote the set of all infinite matrices which map α into itself, the series involved in the transformations being absolutely convergent. It has been proved by G. Köthe and O. Toeplitz [(**4**) 207; (**3**) 313] that $\Sigma(\alpha)$ is a ring if α contains ϕ and is normal, and, in particular, if α is perfect. In this case every matrix which maps α into itself belongs to $\Sigma(\alpha)$ since we can choose the arguments of the coordinates arbitrarily. The space Γ of convergent sequences is not normal whereas $\Sigma(\Gamma)$ is the ring of K-matrices [(**3**) 298]. I believe that this is the only known example of a ring whose elements induce transformations on a sequence space which contains ϕ and fails to satisfy the Köthe–Toeplitz conditions.

In this paper it will be proved that the Köthe–Toeplitz results and the ring $\Sigma(\Gamma)$ are particular cases of a more general result.

THEOREM. *If* (i) $\beta \supseteq \phi$ *and is normal*, (ii) $\beta \subseteq \alpha \subseteq \beta^{**}$, *then* $\Sigma(\alpha)$ *is a ring.*

It is sufficient to prove that $\Sigma(\alpha) \subseteq \Sigma(\alpha^{**})$ [(**3**), 312]. It follows from (ii) that $\alpha^* = \beta^*$ [see (**2**) 275, (10.1, 1) and (10.1, 11)]. Suppose that $A \in \Sigma(\alpha)$; then, by (ii), A maps β into β^{**}, and all the series involved in the transformations are absolutely convergent. Then [(**3**) 298]

$$\Sigma(\alpha) \subseteq \beta \to \beta^{**}.$$

Since $\beta \supseteq \phi$ and is normal, and β^{**} is perfect, then [(**1**) 375]

$$(\beta \to \beta^{**})' = \beta^* \to \beta^* = \Sigma(\beta^*) = \Sigma(\alpha^*).$$

Hence $\Sigma'(\alpha) \subseteq \Sigma(\alpha^*)$. Since α^* is perfect, we have $\Sigma'(\alpha^*) = \Sigma(\alpha^{**})$ [(**3**) 300], and thus we have proved that $\Sigma(\alpha) \subseteq \Sigma(\alpha^{**})$. This proves the theorem.

The conditions of the theorem are satisfied if we take β to be the space Z of null sequences and $\alpha = \Gamma$. We have $Z^{**} = \sigma_\infty$, the space of bounded sequences [(**2**) 277].

We can also apply the theorem taking β to be δ, where δ is the space of all sequences such that, if d_n is the number of non-zero coordinates

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in the first n coordinates, then [(**2**) 274]

$$\lim_{n\to\infty} \frac{d_n}{n} = 0.$$

We have $\delta^{**} = \sigma$, the space of all sequences.

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PRINCIPAL SYSTEMS

By D. G. NORTHCOTT (*Sheffield*) *and* D. REES (*Cambridge*)

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1. Introduction

LET Q be a local ring with maximal ideal $\mathfrak{m}$ and let us make the following definition which is suggested by Macaulay's theory of inverse systems [(**6**) Chap. IV].

DEFINITION. *A proper† ideal* $\mathfrak{a}$ *of* Q *will be called a 'principal system' if, for every* $\mathfrak{m}$*-primary ideal* $\mathfrak{q}$ *containing* $\mathfrak{a}$, *there exists an irreducible‡* $\mathfrak{m}$*-primary ideal* $\mathfrak{q}'$ *satisfying* $\mathfrak{a} \subseteq \mathfrak{q}' \subseteq \mathfrak{q}$.

It seems that this may well prove a fruitful concept, and therefore we have set out in the present paper the basic facts concerning principal systems so far as we have been able to determine them. The most fundamental of these are the following.

(A) *If* $\mathfrak{a}$ *is a principal system and* $b \notin \mathfrak{a}$, *then* $\mathfrak{a}:b$ *is a principal system.*

(B) *If* $\mathfrak{a}:\mathfrak{a}' = \mathfrak{a}$ *and* $\mathfrak{a}':\mathfrak{a} = \mathfrak{a}'$, *then* $\mathfrak{a} \cap \mathfrak{a}'$ *is a principal system when and only when both* $\mathfrak{a}$ *and* $\mathfrak{a}'$ *are principal systems.*

(C) *If* Q *is a homomorphic image of a regular local ring, then every irreducible ideal of* Q *is a principal system.*

(D) *If* Q *is a complete local ring, then every irreducible ideal is a principal system.§*

Before proceeding, let us note the following obvious facts.

First, if $\mathfrak{a}$ *is a principal system, then it is possible to find a decreasing sequence* $\{\mathfrak{q}_n\}$ *of irreducible* $\mathfrak{m}$*-primary ideals such that*

(i) $\bigcap_{n=1}^{\infty} \mathfrak{q}_n = \mathfrak{a}$

and

(ii) *if* $\mathfrak{q}$ *is an* $\mathfrak{m}$*-primary ideal containing* $\mathfrak{a}$, *then* $\mathfrak{q}_n \subseteq \mathfrak{q}$ *for all sufficiently large values of* n.

We shall make considerable use of this property. Conversely, of course, if there exists a decreasing sequence $\{\mathfrak{q}_n\}$ of irreducible $\mathfrak{m}$-primary ideals for which (i) and (ii) hold, then $\mathfrak{a}$ is a principal system.

† A 'proper' ideal is one which is not the whole ring.

‡ An 'irreducible' ideal is one which cannot be expressed as the intersection of two strictly larger ideals.

§ Indeed, from a theorem by Cohen [(**2**) 89, Theorem 15, Corollary 2] (D) is a special case of (C).

Secondly, if $\bar{Q}$ denotes the completion of Q, then a Q-ideal $\mathfrak{a}$ is a principal system when and only when $\bar{Q}\mathfrak{a}$ is a principal system in $\bar{Q}$.

2. Preliminary lemmas

Let R be a Noetherian ring and let $\mathfrak{a}$ be an ideal of R then, if $\mathfrak{a}$ is irreducible, it must be primary. Let us therefore consider a primary ideal $\mathfrak{n}$ belonging (say) to the prime ideal $\mathfrak{p}$. If $\mathfrak{n}$ can be expressed as the intersection of two strictly larger ideals, then it is possible to express $\mathfrak{n}$ as the intersection of two strictly larger $\mathfrak{p}$-primary ideals. It follows that $\mathfrak{n}$ *is irreducible if and only if the corresponding ideal in the ring of quotients of R with respect to $\mathfrak{p}$ is irreducible.* Besides this elementary observation we require three lemmas concerning irreducible ideals. These can all be reduced to the case in which the ring considered is a primary ring† and for this situation the results are well known.‡

LEMMA 1. *Let $\mathfrak{n}$ be an irreducible $\mathfrak{p}$-primary ideal in a Noetherian ring and let $\mathfrak{a}$ be a $\mathfrak{p}$-primary ideal containing $\mathfrak{n}$. Then* $\mathfrak{a} = \mathfrak{n}:(\mathfrak{n}:\mathfrak{a})$.

LEMMA 2. *Let $\mathfrak{n}$ be an irreducible ideal in a Noetherian ring and suppose that b is an element not contained in $\mathfrak{n}$. Then $\mathfrak{n}:b$ is also irreducible.*

LEMMA 3. *Let Q be a local ring with maximal ideal $\mathfrak{m}$ and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. Then $\mathfrak{q}$ is irreducible if and only if the Q-module* $(\mathfrak{q}:\mathfrak{m})/\mathfrak{q}$ *has unit length.*

COROLLARY. *Let $\mathfrak{q}$ be an irreducible $\mathfrak{m}$-primary ideal in the local ring Q and suppose that $\mathfrak{q}'$ strictly contains $\mathfrak{q}$. Then $\mathfrak{q}'$ contains* $\mathfrak{q}:\mathfrak{m}$.

Proof. By the theory of composition series we can find an ideal $\mathfrak{q}''$ so that $\mathfrak{q} \subset \mathfrak{q}'' \subseteq \mathfrak{q}'$ and there is no ideal strictly between $\mathfrak{q}$ and $\mathfrak{q}''$. Then $\mathfrak{m}\mathfrak{q}'' \subseteq \mathfrak{q}$ and therefore $\mathfrak{q} \subset \mathfrak{q}'' \subseteq \mathfrak{q}:\mathfrak{m}$. But, by the lemma, the Q-module $\mathfrak{q}:\mathfrak{m}$ has unit length; consequently $\mathfrak{q}:\mathfrak{m} = \mathfrak{q}'' \subseteq \mathfrak{q}'$ as required.

The next three lemmas are general results concerning local rings. To avoid constant repetition we shall agree that, throughout the rest of the paper, *Q will always denote a local ring with maximal ideal* $\mathfrak{m}$.

LEMMA 4. *Let $\mathfrak{a}$ be an ideal of Q and let b be an element of Q. Then there exists an integer k such that* $(\mathfrak{a}+\mathfrak{m}^n):b \subseteq \mathfrak{a}:b+\mathfrak{m}^{n-k}$ *for all $n > k$.*

Proof. By passing to the ring $Q/\mathfrak{a}$ we see that we can assume that $\mathfrak{a} = (0)$. For this situation, however, the lemma is a special case of a result due to Rees [(**9**) 156, Corollary].

† A primary ring is a Noetherian ring with only one proper prime ideal.
‡ See Krull [(**5**) 33] as a general reference.

LEMMA 5. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of* Q *and let* $\mathfrak{q}$ *be an* $\mathfrak{m}$*-primary ideal containing* $\mathfrak{a} \cap \mathfrak{b}$. *Then*

$$(\mathfrak{a}+\mathfrak{m}^k) \cap (\mathfrak{b}+\mathfrak{m}^k) \subseteq \mathfrak{q}$$

for all sufficiently large values of k.

Proof. Let $\bar{Q}$ be the completion of Q. If $\mathfrak{c}$ denotes a Q-ideal, we shall use $\bar{\mathfrak{c}}$ to denote its extension to $\bar{Q}$. Now by (**10**) 11, Proposition 3,

$$\overline{(\mathfrak{a} \cap \mathfrak{b})} = \bar{\mathfrak{a}} \cap \bar{\mathfrak{b}}$$

and so
$$\bar{\mathfrak{a}} \cap \bar{\mathfrak{b}} \subseteq \bar{\mathfrak{q}}.$$

Furthermore, the ideals

$$(\bar{\mathfrak{a}}+\bar{\mathfrak{m}}^k) \cap (\bar{\mathfrak{b}}+\bar{\mathfrak{m}}^k),$$

for $k = 1, 2,\ldots,$ form a *decreasing* sequence and their intersection is

$$\bigcap_{k=1}^{\infty} (\bar{\mathfrak{a}}+\bar{\mathfrak{m}}^k) \cap \bigcap_{k=1}^{\infty} (\bar{\mathfrak{b}}+\bar{\mathfrak{m}}^k) = \bar{\mathfrak{a}} \cap \bar{\mathfrak{b}}.$$

It follows† that
$$(\bar{\mathfrak{a}}+\bar{\mathfrak{m}}^k) \cap (\bar{\mathfrak{b}}+\bar{\mathfrak{m}}^k) \subseteq \bar{\mathfrak{q}}$$

for all large values of k and now, projecting back on to Q, we obtain the required result.

It will be convenient to use 'dim' as an abbreviation for *dimension*.

LEMMA 6. *Let* $d = \dim Q$ *and suppose that there is a prime ideal* $\mathfrak{p}$ *belonging to* (0) *such that* $\dim \mathfrak{p} < d$. *If now* $a \in \mathfrak{m}$ *and is not a zero-divisor, then at least one of the prime ideals belonging to* (a) *has dimension smaller than* $d-1$.

Proof. Assume the contrary. Then all the prime ideals belonging to (a) have dimension $d-1$. Also $a \notin \mathfrak{p}$ because a is not a zero-divisor. It follows that $(a):\mathfrak{p} = (a)$, and therefore we can choose $p \in \mathfrak{p}$ so that $(a):p = (a)$. But p must be a zero-divisor. Suppose that $px = 0$, where $x \neq 0$. Then $px = 0 \in (a)$, and therefore $x \in (a)$, say $x = ax_1$. Again $0 = px = pax_1$, and therefore $px_1 = 0$ because a is not a zero-divisor. Thus $px_1 = 0 \in (a)$ and hence $x_1 \in (a)$, say $x_1 = ax_2$. We now have $x \in (a^2)$. Proceeding in this way we find that $x \in (a^n)$ for all values of n. But this implies that $x = 0$, and so we have the required contradiction.

3. Principal systems and unmixed ideals

We shall devote this section to showing how the notion of a principal system throws light, from a new direction, on the theory of unmixed ideals.

† Here we use a well-known property of *complete* local rings. See, for example, (**8**) 86, Theorem 1.

LEMMA 7. *Let $\mathfrak{a}$ be a principal system in Q and suppose that $\mathfrak{a}$ is not $\mathfrak{m}$-primary. Then* $\mathfrak{a}:\mathfrak{m} = \mathfrak{a}$.

Proof. Select a decreasing sequence $\{\mathfrak{q}_n\}$ of irreducible $\mathfrak{m}$-primary ideals having properties (i) and (ii) of § 1. Since $\mathfrak{a}$ is not $\mathfrak{m}$-primary, we may suppose that the sequence decreases strictly. Accordingly, by the corollary to Lemma 3,

$$\mathfrak{q}_n:\mathfrak{m} \subseteq \mathfrak{q}_{n-1} \quad (n \geqslant 2),$$

and therefore

$$\mathfrak{a}:\mathfrak{m} = \Big(\bigcap_{n=2}^{\infty} \mathfrak{q}_n\Big):\mathfrak{m} = \bigcap_{n=2}^{\infty} (\mathfrak{q}_n:\mathfrak{m}) \subseteq \bigcap_{n=2}^{\infty} \mathfrak{q}_{n-1} = \mathfrak{a}.$$

The lemma follows.

THEOREM 1. *Let Q be a d-dimensional local ring with the property that every ideal generated by a system of parameters is irreducible. If now $(x_1, x_2,..., x_r)$ is an ideal of dimension $d-r$, then all its prime ideals are $(d-r)$-dimensional.*

Proof. Assume the contrary. Then there is a prime ideal belonging to $(x_1, x_2,..., x_r)$ whose dimension is less than $d-r$. If $\mathfrak{m}$ does not belong to $(x_1, x_2,..., x_r)$, then we can choose $x_{r+1} \in \mathfrak{m}$ so that

$$(x_1,..., x_r):x_{r+1} = (x_1,..., x_r).$$

Applying Lemma 6 to the ring $Q/(x_1,..., x_r)$, we find that $(x_1,..., x_r, x_{r+1})$ has dimension $d-r-1$ and that it possesses a prime ideal of smaller dimension than this. It follows that, for the remainder of the proof, we may assume that $(x_1,..., x_r)$ *has the additional property of possessing* $\mathfrak{m}$ *as an embedded prime ideal.*

Since $\dim(x_1,..., x_r) = d-r$, the set $x_1, x_2,..., x_r$ is a subset of a system $x_1,..., x_r, x_{r+1},..., x_d$ of parameters [(**8**) 64, Theorem 2]. Consequently, if $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal containing $(x_1,..., x_r)$, then, provided that n is large enough,

$$(x_1,..., x_r) \subseteq (x_1,..., x_r, x_{r+1}^n,..., x_d^n) \subseteq \mathfrak{q},$$

and, by hypothesis, $(x_1,..., x_r, x_{r+1}^n,..., x_d^n)$ is irreducible. Thus it has been shown that $(x_1,..., x_r)$ is a principal system and it is clearly not $\mathfrak{m}$-primary. Accordingly, by Lemma 7,

$$(x_1,..., x_r):\mathfrak{m} = (x_1,..., x_r),$$

and this is the required contradiction.

Remarks. At this point it is convenient to make a number of observations concerning regular local rings.

(A) The latter part of the proof of Theorem 1 makes use of the following simple idea. *If a local ring has the property that every ideal generated by*

a system of parameters is irreducible, then every ideal which can be generated by a subset of a system of parameters is a principal system. Now there is a theorem due to Gröbner (**3**) to the effect that, *in a regular local ring, systems of parameters always generate irreducible ideals.* Accordingly, if Q is a d-dimensional regular local ring, then an ideal $(x_1, x_2,..., x_r)$ of dimension $d-r$ must necessarily be a principal system.

(B) Suppose that Q is a d-dimensional regular local ring. By a theorem of Krull's [(**4**) 211, Theorem 11], if $\mathfrak{p}$ is a prime ideal, then

$$\dim \mathfrak{p} + \operatorname{rank} \mathfrak{p} = d.$$

Thus, for Q, the notions of rank and dimension are entirely complementary and therefore *to say that* $(x_1, x_2,..., x_r)$ *is of dimension* $d-r$ *is equivalent to saying that it is an ideal of rank* r.

(C) If we combine Theorem 1 with Gröbner's theorem, it is interesting to note that we obtain the well-known Macaulay–Cohen theorem: *if Q is a d-dimensional regular local ring and $(x_1, x_2,..., x_r)$ is an ideal of dimension $d-r$, then all the prime ideals of $(x_1, x_2,..., x_r)$ have the same dimension.* It will be noticed that the theorem has been stated in terms of dimension rather than rank but, by virtue of (B), the two forms are equivalent.

(D) In a regular local ring Q, let $(x_1, x_2,..., x_r)$ be an ideal of rank r and let $\mathfrak{p}$ be one of its prime ideals. The Macaulay–Cohen theorem shows that $\operatorname{rank} \mathfrak{p} = r$ and therefore, in particular, $\mathfrak{p}$ is a minimal prime ideal of $(x_1, x_2,..., x_r)$. Denote by $Q_\mathfrak{p}$ the ring of quotients of Q with respect to $\mathfrak{p}$. This is a local ring of dimension r and, in it, $Q_\mathfrak{p}(x_1, x_2,..., x_r)$ is an ideal generated by a system of parameters. Further, by a recently announced theorem of Auslander and Buchsbaum (**1**), $Q_\mathfrak{p}$ is a regular ring. Gröbner's theorem now shows that $Q_\mathfrak{p}(x_1, x_2,..., x_r)$ is irreducible, and therefore the $\mathfrak{p}$-primary component of $(x_1, x_2,..., x_r)$ is also irreducible. In brief, then, all the primary components of $(x_1, x_2,..., x_r)$ are irreducible.

We bring together some of these observations in

THEOREM 2. *Let $(x_1, x_2,..., x_r)$ be an ideal of rank r in a regular local ring. Then $(x_1, x_2,..., x_r)$ is a principal system, each of its prime ideals has rank r, and its primary components are all irreducible.*†

4. Some basic properties of principal systems

The simplest general result concerning principal systems is given by

THEOREM 3. *Let $\mathfrak{a}$ be a principal system in a local ring Q and let b be an element not contained in $\mathfrak{a}$. Then $\mathfrak{a}:b$ is a principal system.*

† Cf. Macaulay (**7**) 39–40, which gives a brief survey of the classical background to most of the results in this paper.

Proof. Let $\{\mathfrak{q}_n\}$ be a decreasing sequence of irreducible $\mathfrak{m}$-primary ideals having the properties (i) and (ii) of § 1. Since $b \notin \mathfrak{a}$, we can arrange (by discarding sufficiently many terms at the beginning) that, for each n, $b \notin \mathfrak{q}_n$. Then

$$\mathfrak{a}:b = \bigcap_1^\infty (\mathfrak{q}_n:b),$$

and, by Lemma 2, the $\mathfrak{q}_n:b$ form a decreasing sequence of irreducible $\mathfrak{m}$-primary ideals. Suppose that $\mathfrak{a}:b \subseteq \mathfrak{q}$, where $\mathfrak{q}$ is $\mathfrak{m}$-primary. The proof will be complete if we can show that $\mathfrak{q}_n:b \subseteq \mathfrak{q}$ for some value of n. Now we can find an increasing sequence $\{s_n\}$ of positive integers such that, for all values of n, $\mathfrak{q}_{s_n} \subseteq \mathfrak{a}+\mathfrak{m}^n$. Furthermore, by Lemma 4, there exists an integer k such that

$$(\mathfrak{a}+\mathfrak{m}^n):b \subseteq \mathfrak{a}:b+\mathfrak{m}^{n-k}$$

for $n > k$. Consequently, for n large enough,

$$\mathfrak{q}_{s_n}:b \subseteq (\mathfrak{a}+\mathfrak{m}^n):b \subseteq \mathfrak{a}:b+\mathfrak{m}^{n-k} \subseteq \mathfrak{q}.$$

This completes the proof.

The next result is needed, on this occasion, in an auxiliary capacity, but there are strong indications that it is likely to occupy a central position in generalizations of Macaulay's theory. In any case, it seems to have an intrinsic interest and it would be desirable to have a simple *ad hoc* proof. The one given here uses the full force of Cohen's structure theorems for complete local rings.

THEOREM 4. *Every primary ring R is a homomorphic image of a primary ring with an irreducible zero ideal.*

Proof. R is a special kind of complete local ring. Hence, by a theorem of Cohen's [(**2**) 89, Theorem 15, Corollary 2], it is a homomorphic image of a regular local ring Q and the kernel of the homomorphism will be an $\mathfrak{m}$-primary ideal $\mathfrak{q}$. Choose, for Q, a system $x_1, x_2,..., x_d$ of parameters so that $(x_1, x_2,..., x_d) \subseteq \mathfrak{q}$. By Gröbner's theorem, $(x_1, x_2,..., x_d)$ is an irreducible ideal. Since $R = Q/\mathfrak{q}$ is a homomorphic image of

$$Q/(x_1, x_2,..., x_d),$$

this completes the proof.

LEMMA 8. *Let $\mathfrak{q}_1$, $\mathfrak{q}_2$ be irreducible $\mathfrak{m}$-primary ideals in a local ring Q and suppose that $\alpha_1 \in \mathfrak{q}_1$, $\alpha_2 \in \mathfrak{q}_2$. Then there exists an irreducible $\mathfrak{m}$-primary ideal $\mathfrak{q}$ containing $\mathfrak{q}_1 \cap \mathfrak{q}_2$ and such that*

$$\mathfrak{q}:\alpha_1 = \mathfrak{q}_2:\alpha_1, \qquad \mathfrak{q}:\alpha_2 = \mathfrak{q}_1:\alpha_2.$$

Proof. We shall first reduce the lemma to the special case in which Q is a primary ring whose zero ideal is irreducible. Put

$$Q' = Q/(\mathfrak{q}_1 \cap \mathfrak{q}_2), \qquad \mathfrak{q}'_i = \mathfrak{q}_i/(\mathfrak{q}_1 \cap \mathfrak{q}_2)$$

and let α'_i be the residue of α_i to modulus $\mathfrak{q}_1 \cap \mathfrak{q}_2$. Then $\mathfrak{q}'_1, \mathfrak{q}'_2$ are irreducible ideals, $\alpha'_i \in \mathfrak{q}'_i$ and $\mathfrak{q}'_1 \cap \mathfrak{q}'_2 = (0)$. Suppose now that we can find an irreducible ideal $\mathfrak{q}'$ for which

$$\mathfrak{q}'\!:\!\alpha'_1 = \mathfrak{q}'_2\!:\!\alpha'_1, \qquad \mathfrak{q}'\!:\!\alpha'_2 = \mathfrak{q}'_1\!:\!\alpha'_2.$$

Then the inverse image of $\mathfrak{q}'$ in Q will have all the required properties. Hence, for the purposes of the proof, we may suppose that Q is a primary ring.

Appealing now to Theorem 4 we see that we may suppose that $Q = Q^*/\mathfrak{a}^*$, where Q^* is a primary ring whose zero ideal is irreducible and where $\mathfrak{a}^*$ is a Q^*-ideal. Let $\mathfrak{q}^*_i$ be the ideal of Q^* which corresponds to $\mathfrak{q}_i$ and let α^*_i be a representative of α_i. Then $\mathfrak{q}^*_i$ is irreducible. Furthermore, if we can find an irreducible ideal $\mathfrak{q}^*$ containing $\mathfrak{q}^*_1 \cap \mathfrak{q}^*_2$ and such that

$$\mathfrak{q}^*\!:\!\alpha^*_1 = \mathfrak{q}^*_2\!:\!\alpha^*_1, \qquad \mathfrak{q}^*\!:\!\alpha^*_2 = \mathfrak{q}^*_1\!:\!\alpha^*_2,$$

then $\mathfrak{q}^*/\mathfrak{a}^*$ will have all the properties we need. Our combined remarks show that it will be sufficient to prove the lemma when Q is a primary ring whose zero ideal is irreducible and, from now on, we shall suppose that we have this situation.†

Since $\mathfrak{q}_i$ is irreducible, $(0)\!:\!\mathfrak{q}_i$ is a principal ideal say $(0)\!:\!\mathfrak{q}_i = (\beta_i)$. Put $\beta = \beta_1+\beta_2$ and $\mathfrak{q} = (0)\!:\!\beta$. Then $\mathfrak{q}$ is irreducible. Also

$$\mathfrak{q} = (0)\!:\!(\beta) \supseteq (0)\!:\!\big((\beta_1)+(\beta_2)\big) = \big((0)\!:\!\beta_1)\big) \cap \big((0)\!:\!\beta_2\big) = \mathfrak{q}_1 \cap \mathfrak{q}_2.$$

Now $\alpha_1 \in \mathfrak{q}_1$, $\beta_1 \in (0)\!:\!\mathfrak{q}_1$. Consequently $\alpha_1\beta_1 = 0$ and therefore

$$\alpha_1\beta = \alpha_1\beta_2.$$

Thus

$$\begin{aligned}\mathfrak{q}_2\!:\!\alpha_1 = \big((0)\!:\!\beta_2\big)\!:\!\alpha_1 &= (0)\!:\!\alpha_1\beta_2 = (0)\!:\!\alpha_1\beta \\ &= \big((0)\!:\!\beta\big)\!:\!\alpha_1 = \mathfrak{q}\!:\!\alpha_1\end{aligned}$$

and similarly $\mathfrak{q}_1\!:\!\alpha_2 = \mathfrak{q}\!:\!\alpha_2$. This completes the proof.

THEOREM 5. *Let $\mathfrak{a}$ and $\mathfrak{a}'$ be proper ideals in a local ring Q such that $\mathfrak{a}\!:\!\mathfrak{a}' = \mathfrak{a}$ and $\mathfrak{a}'\!:\!\mathfrak{a} = \mathfrak{a}'$. Then $\mathfrak{a} \cap \mathfrak{a}'$ is a principal system when and only when both $\mathfrak{a}$ and $\mathfrak{a}'$ are principal systems.*

Proof. Choose $\alpha' \in \mathfrak{a}'$ so that $\mathfrak{a}\!:\!\alpha' = \mathfrak{a}$, and $\alpha \in \mathfrak{a}$ so that $\mathfrak{a}'\!:\!\alpha = \mathfrak{a}'$.

† We make free use of the properties of such rings. The reader will find the relevant facts set out in Krull (**5**) 30–33.

Then $$(\mathfrak{a} \cap \mathfrak{a}'):\alpha = \mathfrak{a}', \qquad (\mathfrak{a} \cap \mathfrak{a}'):\alpha' = \mathfrak{a}.$$
Accordingly, by Theorem 3, if $\mathfrak{a} \cap \mathfrak{a}'$ is a principal system, then so are $\mathfrak{a}$ and $\mathfrak{a}'$.

From now on we assume that $\mathfrak{a}$ and $\mathfrak{a}'$ are principal systems. Choose a decreasing sequence $\{\mathfrak{q}_n\}$ of irreducible $\mathfrak{m}$-primary ideals so that (i) and (ii) of § 1 are satisfied and, for the ideal $\mathfrak{a}'$, choose a sequence $\{\mathfrak{q}'_n\}$ with similar properties. Applying Lemma 8 to $\mathfrak{q}_n$, $\mathfrak{q}'_n$ and the elements α, α', we see that there exists an irreducible $\mathfrak{m}$-primary ideal $\bar{\mathfrak{q}}_n$ such that
$$\bar{\mathfrak{q}}_n \supseteq \mathfrak{q}_n \cap \mathfrak{q}'_n \supseteq \mathfrak{a} \cap \mathfrak{a}', \qquad \bar{\mathfrak{q}}_n:\alpha = \mathfrak{q}'_n:\alpha, \qquad \bar{\mathfrak{q}}_n:\alpha' = \mathfrak{q}_n:\alpha'.$$
Of course, the $\bar{\mathfrak{q}}_n$ need not form a decreasing sequence. Let $\mathfrak{q}^*$ be an $\mathfrak{m}$-primary ideal containing $\mathfrak{a} \cap \mathfrak{a}'$. To complete the proof we need only show that $\bar{\mathfrak{q}}_n \subseteq \mathfrak{q}^*$ for some value of n.

By Lemma 5, we can choose r so that
$$(\mathfrak{a}+\mathfrak{m}^r) \cap (\mathfrak{a}'+\mathfrak{m}^r) \subseteq \mathfrak{q}^*.$$
Further, since $\mathfrak{a}:\alpha' = \mathfrak{a}$ and $\mathfrak{a}':\alpha = \mathfrak{a}'$, we see, by Lemma 4, that, if we choose s large enough,
$$(\mathfrak{a}+\mathfrak{m}^s):\alpha' \subseteq \mathfrak{a}+\mathfrak{m}^r \quad \text{and} \quad (\mathfrak{a}'+\mathfrak{m}^s):\alpha \subseteq \mathfrak{a}'+\mathfrak{m}^r.$$
Having chosen s, now choose n so that $\mathfrak{q}_n \subseteq \mathfrak{a}+\mathfrak{m}^s$ and $\mathfrak{q}'_n \subseteq \mathfrak{a}'+\mathfrak{m}^s$; then
$$\begin{aligned}\bar{\mathfrak{q}}_n &\subseteq (\bar{\mathfrak{q}}_n:\alpha') \cap (\bar{\mathfrak{q}}_n:\alpha) = (\mathfrak{q}_n:\alpha') \cap (\mathfrak{q}'_n:\alpha) \\ &\subseteq \{(\mathfrak{a}+\mathfrak{m}^s):\alpha'\} \cap \{(\mathfrak{a}'+\mathfrak{m}^s):\alpha\} \subseteq (\mathfrak{a}+\mathfrak{m}^r) \cap (\mathfrak{a}'+\mathfrak{m}^r) \\ &\subseteq \mathfrak{q}^*.\end{aligned}$$
This proves the theorem.

LEMMA 9. *Let Q be a homomorphic image of a regular local ring and let $\mathfrak{n}$ be a primary ideal of Q. Then $\mathfrak{n}$ is the intersection of a finite number of principal systems.*

Proof. Without loss of generality we may suppose that Q itself is regular. Let $\mathfrak{n}$ be $\mathfrak{p}$-primary and of rank r. Then we can choose r elements $x_1, x_2,..., x_r$ in $\mathfrak{n}$ so that $(x_1, x_2,..., x_r)$ is an ideal of rank r.† Clearly $\mathfrak{p}$ belongs to $(x_1, x_2,.., x_r)$ and, if $\mathfrak{n}_0$ is the $\mathfrak{p}$-primary component, then $\mathfrak{n}_0 \subseteq \mathfrak{n}$. Moreover, by virtue of Theorems 2 and 5, $\mathfrak{n}_0$ *is both an irreducible ideal and a principal system.*

Put $$\mathfrak{n}_0:\mathfrak{n} = \mathfrak{a} = (a_1, a_2,..., a_s), \text{ say.}$$
Then, by Lemma 1,
$$\mathfrak{n} = \mathfrak{n}_0:(\mathfrak{n}_0:\mathfrak{n}) = \mathfrak{n}_0:\mathfrak{a} = \bigcap_{i=1}^{s} (\mathfrak{n}_0:a_i).$$

† See, for example, (8) 61, Theorem 8.

But, by Theorem 3, $\mathfrak{n}_0 : \mathfrak{a}_i$ is either the whole ring, in which case it can be ignored, or else it is a principal system. This establishes the lemma.

THEOREM 6. *Let Q be a homomorphic image of a regular local ring. Then every ideal of Q is the intersection of a finite number of principal systems. Further, every irreducible ideal is a principal system.*

Proof. The first assertion follows at once from Lemma 9. Let $\mathfrak{a}$ be an irreducible ideal. Then we can write

$$\mathfrak{a} = \mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \ldots \cap \mathfrak{a}_s,$$

where each $\mathfrak{a}_i$ is a principal system and the integer s is minimal. We must now have $s = 1$. For, if $s \geqslant 2$, then $\mathfrak{a}_1$ and $\mathfrak{a}_2 \cap \ldots \cap \mathfrak{a}_s$ both strictly contain $\mathfrak{a}$, and this is impossible because $\mathfrak{a}$ is irreducible.

Since every complete local ring is a homomorphic image of a regular local ring, we have, in particular, the corollary:

COROLLARY. *In a complete local ring every irreducible ideal is a principal system.*

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A NOTE ON MATRIX POLYNOMIALS

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1. LET $\phi(x)$ be a polynomial and $\mathbf{A}$ a square matrix. It is evident that, if $\mathbf{A}$ is diagonable (i.e. similar to a diagonal matrix), then so is $\phi(\mathbf{A})$. Equally, if $\mathbf{A}$ is normal (i.e. if it commutes with its transposed conjugate), then so is $\phi(\mathbf{A})$. Our object in § 2 is to establish conditions sufficient to ensure that the converse inferences should be valid.

Suppose, now, that $\mathbf{A}$ can be expressed as a polynomial in $\phi(\mathbf{A})$. Then the diagonability or normality of $\phi(\mathbf{A})$ imply those of $\mathbf{A}$. This suggests the problem of determining under what conditions there exists a polynomial $p(x)$ such that $p\{\phi(\mathbf{A})\} = \mathbf{A}$. We shall solve this problem in § 3 by means of a general result on polynomial congruences. Finally, in § 4, we shall give generalizations of earlier theorems in which the polynomial $\phi(x)$ is replaced by a regular function.

2. THEOREM 1. *Let $\mathbf{A}$ be a complex $n \times n$ matrix with characteristic roots $\omega_1,..., \omega_n$. Let $\phi(x)$ be any polynomial and $f(x)$ a polynomial such that $f(\mathbf{A}) = \mathbf{O}$.*

(i) *Suppose that*

$$\phi'(\omega_r) \neq 0 \quad \textit{whenever } f'(\omega_r) = 0.\dagger \tag{1}$$

Then, if $\phi(\mathbf{A})$ is diagonable, so is $\mathbf{A}$. In particular, this conclusion is valid if no characteristic root of $\mathbf{A}$ is a zero of $\phi'(x)$.

(ii) *Suppose that, in addition to* (1), *we also have*

$$\phi(\omega_r) \neq \phi(\omega_s) \quad \textit{whenever } \omega_r \neq \omega_s. \tag{2}$$

Then, if $\phi(\mathbf{A})$ is normal, so is $\mathbf{A}$.

To prove (i), assume that (1) is satisfied. Denote by $R(\mathbf{A})$ the rank of $\mathbf{A}$ and by $m(\mathbf{A};\omega)$ the multiplicity of ω as characteristic root of $\mathbf{A}$.

Suppose, in the first place, that $f'(\omega_1) = 0$, so that $\phi'(\omega_1) \neq 0$. We have

$$\phi(x) - \phi(\omega_1) = (x-\omega_1)\psi(x)$$

for a certain polynomial $\psi(x)$. Hence

$$\phi(\mathbf{A}) - \phi(\omega_1)\mathbf{I} = (\mathbf{A}-\omega_1 \mathbf{I})\psi(\mathbf{A}). \tag{3}$$

† If, for a given ϕ, at least one polynomial $f(x)$ annihilating $\mathbf{A}$ satisfies (1), then so does the minimum polynomial of $\mathbf{A}$.

The characteristic roots of $\psi(\mathbf{A})$ are $\psi(\omega_1),\ldots, \psi(\omega_n)$. Therefore, since $\psi(\omega_1) = \phi'(\omega_1)$, $m\{\psi(\mathbf{A}); 0\}$ is equal to the number of r such that

$$1 \leqslant r \leqslant n, \qquad \phi(\omega_r) = \phi(\omega_1), \qquad \omega_r \neq \omega_1;$$

thus

$$m\{\psi(\mathbf{A}); 0\} = m\{\phi(\mathbf{A}); \phi(\omega_1)\} - m(\mathbf{A}; \omega_1). \tag{4}$$

If $\phi(\mathbf{A})$ is diagonable, then, using (3) and (4), we obtain

$$\begin{aligned} n - m\{\phi(\mathbf{A}); \phi(\omega_1)\} &= R\{\phi(\mathbf{A}) - \phi(\omega_1)\mathbf{I}\} = R\{(\mathbf{A}-\omega_1\mathbf{I})\psi(\mathbf{A})\} \\ &\geqslant R(\mathbf{A}-\omega_1\mathbf{I}) + R\{\psi(\mathbf{A})\} - n \geqslant R(\mathbf{A}-\omega_1\mathbf{I}) + [n - m\{\psi(\mathbf{A}); 0\}] - n \\ &= R(\mathbf{A}-\omega_1\mathbf{I}) - m\{\phi(\mathbf{A}); \phi(\omega_1)\} + m(\mathbf{A}; \omega_1) \geqslant n - m\{\phi(\mathbf{A}); \phi(\omega_1)\}. \end{aligned}$$

Hence

$$R(\mathbf{A}-\omega_1\mathbf{I}) = n - m(\mathbf{A}; \omega_1). \tag{5}$$

Now let $f'(\omega_1) \neq 0$. Then, since $f(\omega_1) = 0$, it follows that ω_1 is a simple zero of $f(x)$ and so of the minimum polynomial of $\mathbf{A}$. Therefore (5) is again valid. But ω_1 is an arbitrary characteristic root of $\mathbf{A}$; hence $\mathbf{A}$ is diagonable, and the proof of (i) is complete.†

Suppose, next, that both (1) and (2) are satisfied. In view of (2) we know that the principal idempotent elements of $\phi(\mathbf{A})$ are the same as those of $\mathbf{A}$.‡ Now, a matrix is normal if and only if it is diagonable and all its principal idempotent elements are hermitian. Hence, if $\phi(\mathbf{A})$ is normal, all principal idempotent elements of $\mathbf{A}$ are hermitian. Moreover, $\mathbf{A}$ is diagonable by (i). Therefore it is normal, and (ii) is proved.

It is easily demonstrated by simple examples that the restrictions imposed on $\mathbf{A}$ and $\phi(x)$ in (i) and (ii) cannot be dispensed with. Thus, if

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \phi(x) = x^2,$$

then $\phi(\mathbf{A})$ is diagonable whereas $\mathbf{A}$ is not; in this case (1) is not satisfied for any $f(x)$ annihilating $\mathbf{A}$. Again, if

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \phi(x) = 0,$$

then $\phi(\mathbf{A})$ is normal whereas $\mathbf{A}$ is not; here condition (2) is violated.

3. Our next object is to determine necessary and sufficient conditions for $\mathbf{A}$ to be expressible in the form $p\{\phi(\mathbf{A})\}$, where $p(x)$ is a suitable polynomial. The case

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad f(x) = x^2 - 1, \qquad \phi(x) = x^2$$

† An alternative procedure is to make use of the classical canonical form.

‡ See e.g. Wedderburn, *Lectures on matrices* (New York, 1934), 30.

shows that the hypothesis of Theorem 1 (i) does not furnish a sufficient condition. To settle the problem, we need to establish the following result on polynomial congruences.

THEOREM 2. *Let $f(x)$, $\phi(x)$ be polynomials with coefficients in a field E; and let E' be an extension field of E in which $f(x)$ factorizes completely, say*

$$f(x) = \prod_{r=1}^{m} (x-\alpha_r)^{k_r},$$

where $\alpha_1,\ldots, \alpha_m$ are distinct and $k_1,\ldots, k_m \geqslant 1$. Then there exists a polynomial $p(x) \in E[x]$ such that

$$p\{\phi(x)\} \equiv x \pmod{f(x)}$$

if and only if

$$\phi(\alpha_r) \neq \phi(\alpha_s) \quad (1 \leqslant r < s \leqslant m) \tag{6}$$

and

$$\phi'(\alpha_r) \neq 0 \quad \textit{whenever } k_r > 1. \tag{7}$$

Suppose, in the first place, that

$$p\{\phi(x)\} = x+f(x)p_1(x), \tag{8}$$

where $p(x)$, $p_1(x) \in E[x]$. Then, for $1 \leqslant r < s \leqslant m$,

$$p\{\phi(\alpha_r)\} = \alpha_r \neq \alpha_s = p\{\phi(\alpha_s)\},$$

so that (6) is satisfied. Furthermore, if $k_r > 1$, then $f'(\alpha_r) = 0$, and therefore $p'\{\phi(\alpha_r)\}\phi'(\alpha_r) = 1$; hence (7) follows.

Next, suppose that (6) and (7) are satisfied. It is to be understood that the coefficients of all polynomials introduced below are elements of E' unless the contrary is stated. For every value of r in the range $1 \leqslant r \leqslant m$ we shall determine a polynomial $q_r(y)$ such that

$$q_r\{\phi(x)\} \equiv x \pmod{(x-\alpha_r)^{k_r}}. \tag{9}$$

If $k_r = 1$, $\phi(\alpha_r) = 0$, we may take $q_r(y) = y+\alpha_r$. If $k_r = 1$, $\phi(\alpha_r) \neq 0$, we may take $q_r(y) = \alpha_r y/\phi(\alpha_r)$. Next, let $k_r \geqslant 2$. Then, for $1 \leqslant s < k_r$,

$$\{\phi(x)-\phi(\alpha_r)\}^s \equiv \sum_{t=s}^{k_r-1} c_{rst}(x-\alpha_r)^t \pmod{(x-\alpha_r)^{k_r}}, \tag{10}$$

where the c's belong to E' and, by (7), $c_{rss} = \{\phi'(\alpha_r)\}^s \neq 0$. Using (10) successively for $s = 1, 2,\ldots, k_r-1$, we infer the existence of a polynomial $\psi_r(y)$ such that

$$x-\alpha_r \equiv \psi_r\{\phi(x)\} \pmod{(x-\alpha_r)^{k_r}},$$

and it follows that $q_r(y) = \psi_r(y)+\alpha_r$ satisfies (9).

In view of (6) it is now possible to choose a polynomial $q(y)$ such that

$$q(y) \equiv q_r(y) \pmod{\{y-\phi(\alpha_r)\}^{k_r}} \quad (r = 1,\ldots, m).$$

Hence $\quad q\{\phi(x)\} \equiv q_r\{\phi(x)\} \pmod{(x-\alpha_r)^{k_r}} \quad (r = 1,..., m),$

and so, by (9), $\quad q\{\phi(x)\} \equiv x \pmod{f(x)}.$

Thus there exist polynomials $q(x)$, $q_1(x) \in E'[x]$ such that

$$q\{\phi(x)\} = x+f(x)q_1(x). \tag{11}$$

But (11) is a system of linear equations for the coefficients of $q(x)$ and $q_1(x)$, and the coefficients of this system are elements of E. Hence, since the system is soluble, it is soluble in E, i.e. there exist polynomials $p(x)$, $p_1(x) \in E[x]$ satisfying (8). This completes the proof of Theorem 2.

THEOREM 3. *Let $\phi(x)$ be a polynomial, and denote by*

$$f(x) = \prod_{r=1}^{m} (x-\alpha_r)^{k_r}$$

(where $\alpha_1,..., \alpha_r$ are distinct and $k_1,..., k_m \geqslant 1$) the minimum polynomial of the matrix $\mathbf{A}$. *Then there exists a polynomial $p(x)$ such that*

$$p\{\phi(\mathbf{A})\} = \mathbf{A} \tag{12}$$

if and only if the relations (6) *and* (7) *are satisfied.*

This result follows at once from Theorem 2 since (12) is equivalent to the congruence $\quad p\{\phi(x)\} \equiv x \pmod{f(x)}.$

If the coefficients of $\phi(x)$ and the characteristic roots of $\mathbf{A}$ are real and if a polynomial $p(x)$ satisfying (12) exists, then it can be chosen in such a way that its coefficients are real.

It may also be of interest to note that Theorem 1 (ii) is an immediate consequence of Theorem 3.

4. Let $\chi(z)$ denote a function regular in the circle $|z| < \rho$, and let its expansion as a power series be given by

$$\chi(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu.$$

If all characteristic roots of $\mathbf{A}$ are, in modulus, less than ρ, then $\chi(\mathbf{A})$ is defined as the sum of the (necessarily convergent) matrix power series

$$\chi(\mathbf{A}) = \sum_{\nu=0}^{\infty} c_\nu \mathbf{A}^\nu.$$

It is now easy to see that Theorems 1 and 3 continue to hold when the polynomial $\phi(x)$ is replaced by the regular function $\chi(z)$. To verify this we need merely recall that there exists a polynomial $\phi_1(x)$, whose

coefficients depend on $\chi(z)$ and $\mathbf{A}$, such that $\chi(\mathbf{A}) = \phi_1(\mathbf{A})$ and

$$\chi'(\alpha) = \phi_1'(\alpha)$$

for every multiple zero α of the minimum polynomial of $\mathbf{A}$.

From the extended form of Theorem 1 it follows, in particular, that, if $\exp \mathbf{A}$ is diagonable, then so is $\mathbf{A}$. Again, if $\exp \mathbf{A}$ is normal and no two characteristic roots of $\mathbf{A}$ differ by an integral multiple of $2\pi i$, then $\mathbf{A}$ is also normal.

ON THE H-FUNCTIONS OF S. CHANDRASEKHAR

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1. Introduction

CHANDRASEKHAR's H-functions are solutions of integral equations of the form

$$\frac{1}{H(\mu)} = 1-\mu\int\limits_0^1 \frac{\Psi(x)H(x)}{\mu+x}\,dx, \tag{1.1}$$

where $\Psi(x)$ is non-negative for $0 \leqslant x \leqslant 1$ and satisfies the condition

$$\int\limits_0^1 \Psi(x)\,dx \leqslant \tfrac{1}{2}. \tag{1.2}$$

The case of equality in (1.2) is called the *conservative case.*

In Chandrasekhar's introductory papers (**1**) and (**2**) the treatment was non-rigorous, and the only rigorous treatment was a very general one by M. M. Crum in (**4**). In (**3**) there is a modified version of (**4**).

Crum's argument, even as given in (**3**), is not at all easy to follow, and some simpler rigorous treatment is needed for the benefit of scientists working with transfer problems in which H-functions occur. This paper is an attempt to supply this need.

2. The characteristic function

In most practical cases $\Psi(x)$ is a polynomial, and it is here that it is helpful to assume more than Crum does. We shall assume that

$\Psi(z)$ *is real and non-negative for* $0 \leqslant z \leqslant 1$; *that every point of the closed interval* $(-1, 1)$ *lies in a domain* D *of the z-plane in which* $\Psi(z)$ *is regular*; *and that* $\Psi(z)$ *satisfies the condition* (1.2).

Since every point of the interval $(-1, 1)$ is an interior point of D, $\Psi(z)$ is continuous in the interval and bounded in the neighbourhood of any point of the interval.

The solution of (1.1) depends upon the characteristic function

$$T(\mu) = 1-2\mu^2\int\limits_0^1 \frac{\Psi(x)}{\mu^2-x^2}\,dx = 1-2\int\limits_0^1 \Psi(x)\,dx-2\int\limits_0^1 \frac{x^2\Psi(x)}{\mu^2-x^2}\,dx, \tag{2.1}$$

and upon the *characteristic equation*

$$T(\mu) = 0. \tag{2.2}$$

The function $T(\mu)$ is an even function of μ which is regular in the μ-plane cut along $(-1, 1)$. From the right-hand member of (2.1) it is seen that, for large μ,

$$T(\mu) = 1-2\int\limits_0^1 \Psi(x)\,dx + O(\mu^{-2}). \tag{2.3}$$

Since $\Psi(z)$ can be written in the form

$$\Psi(z) = \psi_1(z^2)+z\,\psi_2(z^2), \tag{2.4}$$

where each $\psi_r(z^2)$ is an even function of z which is regular in D, therefore, for μ in D but not on the cut along $(-1, 1)$, $T(\mu)$ can be written

$$T(\mu) = 1-2\mu^2\int\limits_0^1 \{\psi_1(x^2)-\psi_1(\mu^2)+x[\psi_2(x^2)-\psi_2(\mu^2)]\}\frac{dx}{\mu^2-x^2}-$$

$$-\mu\psi_1(\mu^2)\ln\frac{\mu+1}{\mu-1}-\mu^2\psi_2(\mu^2)\ln\frac{\mu^2}{\mu^2-1}, \tag{2.5}$$

where the logarithms are the principal values. The integral on the right of (2.5) represents a function regular in D which has a double zero at $\mu = 0$, and it is therefore seen that

$$T(\mu) = 1+O(|\mu|) \quad \text{as } |\mu| \to 0 \tag{2.6}$$

uniformly in $\arg\mu$ in the cut plane.

With regard to the roots of (2.2), three cases have to be distinguished:

(i) the conservative case when $T(\mu)$ has a double zero at infinity;

(ii) the non-conservative case in which $\lim\limits_{\mu\to 1+0} T(\mu) < 0$, when $T(\mu)$ has simple zeros $\mu = \pm k^{-1}$ $(0 < k < 1)$;†

(iii) the non-conservative case in which $\lim\limits_{\mu\to 1+0} T(\mu) > 0$, when $T(\mu)$ has no zeros in the cut plane.

We shall refer to these cases by numbers. The proof given in (**3**) that these are the only zeros in the cut plane is quite simple.

† If μ is real, $T(\mu)$ is a steadily increasing function of μ in $(1, \infty)$ and, by (1.2) and (2.3), $\lim\limits_{\mu\to\infty} T(\mu) > 0$. Hence there is one zero in $(1, \infty)$ if $\lim\limits_{\mu\to 1+0} T(\mu) < 0$.

3. The functional relation

Although not explicitly stated, the following theorem is proved in (**3**):

THEOREM I. *If $H(\mu)$ is any solution of* (1.1) *which is continuous in the interval $0 \leqslant \mu \leqslant 1$, then $1/H(\mu)$ is regular in the μ-plane cut along $(-1, 0)$. In the μ-plane cut along $(-1, 1)$, $H(\mu)$ satisfies the functional relation*

$$H(\mu)H(-\mu) = 1/T(\mu). \tag{3.1}$$

In case (i), *$H(\mu)$ is regular in the μ-plane cut along $(-1, 0)$ apart from a simple pole at infinity; in case* (ii), *$H(\mu)$ is regular in the cut plane apart from a simple pole at either $\mu = k^{-1}$ or $\mu = -k^{-1}$; in case* (iii), *$H(\mu)$ is regular in the cut plane.*

The analytic properties of $1/H(\mu)$ follow at once from (1.1) and the proof of the relation (3.1) is quite simple [see (**4**) 246, or (**3**) 116]. The analytic properties of $H(\mu)$ follow from those of $1/H(\mu)$ and from the relation (3.1) since zeros of $1/H(\mu)$ can occur only at the zeros of $T(\mu)$.

4. The solution of the H-equation

I shall prove the following theorem:

THEOREM II. *The integral equation* (1.1) *has a solution $H(\mu)$ which is regular in the μ-plane cut along $(-1, 0)$ apart from a simple pole at infinity in case* (i) *and a simple pole at $\mu = -1/k$ in case* (ii). *For* $\operatorname{re}\mu > 0$, *$H(\mu)$ is given by*

$$H(\mu) = \exp\left\{\frac{\mu}{2\pi i}\int\limits_{-i\infty}^{i\infty} \frac{\ln T(z)}{z^2-\mu^2}\,dz\right\}. \tag{4.1}$$

In cases (i) *and* (iii) *this is the only solution of* (1.1) *which is continuous in the interval $0 \leqslant \mu \leqslant 1$. In case* (ii) *the function*

$$H_1(\mu) = \frac{1+k\mu}{1-k\mu}H(\mu) \tag{4.2}$$

is also such a solution, and there are no other solutions.

We shall first obtain the integral (4.1) by a method suggested by the Wiener–Hopf solution of the integral equation

$$J(\tau) = \tfrac{1}{2}\chi\int\limits_0^\infty J(t)E_1(|t-\tau|)\,dt,$$

which gives rise to the H-functions in the particular case $\Psi(x) = \frac{1}{2}\chi$. We shall then verify that (4.1) is a solution of (1.1) and we shall finally consider the uniqueness.

4.1. Let

$$K(s) = \left\{\begin{array}{ll} \dfrac{s^2-1}{s^2}T(s^{-1}) & (\text{case (i)}) \\ \dfrac{s^2-1}{s^2-k^2}T(s^{-1}) & (\text{case (ii)}) \\ T(s^{-1}) & (\text{case (iii)}) \end{array}\right\}. \tag{4.3}$$

By § 2, $K(s)$ is regular and not zero in the s-plane cut along $(-\infty, -1)$ and $(1, \infty)$. In this cut plane, by (2.6),

$$K(s) = 1+O(|s|^{-1}) \quad \text{as } |s| \to \infty, \tag{4.4}$$

uniformly in $\arg s$. From (4.3), (2.1), and (1.2) it is easily seen that $K(it) > 0$ for $-\infty < t < \infty$, and it follows that $\ln K(s)$ can be defined to be that branch which is such that

$$\ln K(s) = O(|s|^{-1}) \quad \text{as } |s| \to \infty \tag{4.5}$$

uniformly in $\arg s$. This branch is regular in the cut plane.

By Cauchy's integral theorem we can write

$$K(s) = K^-(s)/K^+(s), \tag{4.6}$$

where $|\operatorname{re} s| < b$ $(0 < b < 1)$ and

$$\ln K^-(s) = \frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty} \frac{\ln K(z)}{z-s}\,dz, \tag{4.7}$$

$$\ln K^+(s) = \frac{1}{2\pi i}\int\limits_{-b-i\infty}^{-b+i\infty} \frac{\ln K(z)}{z-s}\,dz. \tag{4.8}$$

Both these integrals exist in virtue of (4.5). Again by (4.5), the line of integration for $\ln K^+(s)$ can be deformed into a loop starting from infinity below the cut along $(-\infty, -1)$, passing round the branch-point at -1 and returning to infinity above the cut. With the usual notation

$$\ln K^+(s) = \frac{1}{2\pi i}\int\limits_{-\infty}^{(-1,+)} \frac{\ln K(z)}{z-s}\,dz. \tag{4.9}$$

From this it follows that $\ln K^+(s)$ is regular in the s-plane cut along $(-\infty, -1)$ and that $K^+(s)$ is regular and not zero in this cut plane. Similarly $K^-(s)$ is regular and not zero in the s-plane cut along $(1, \infty)$.

By replacing s and z by $-s$ and $-z$ in (4.8), we see that, if $\operatorname{re} s < b$,

$$\ln K^+(-s) = -\ln K^-(s)$$

and hence that

$$K^+(-s)K^-(s) = 1. \tag{4.10}$$

This holds, by analytic continuation, in the s-plane cut along $(1, \infty)$.

Finally, consider $K^+(s)$ as $|s| \to \infty$ in the plane cut along $(-\infty, -1)$. Let $s = \rho e^{i\phi}$, where $|\phi| \leqslant \frac{1}{2}\pi$. In (4.9) we can take the contour to be the two sides of the cut together with an indentation of radius η about $s = -1$. From (2.5) and (4.3) it is seen that $K(s)$ is at most $O(\ln \eta)$ on the indentation, and hence that the integral round the indentation is $O(\eta \ln\ln \eta)$. On the sides of the cut $z = -x$ and

$$|z-s| \geqslant \rho, \qquad |z-s| \geqslant x.$$

Hence, on choosing η sufficiently small and then using (4.5), we have

$$\begin{aligned} |\ln K^+(s)| &\leqslant O\left(\frac{1}{\rho}\int\limits_{1+\eta}^{\rho}\frac{dx}{x}\right)+O\left(\int\limits_{\rho}^{\infty}\frac{dx}{x^2}\right)+o(1) \\ &= o(1) \quad \text{as } \rho\to\infty, \end{aligned}$$

uniformly in $\arg s$. Thus

$$K^+(s) \to 1 \quad \text{as } |s| \to \infty \quad (|\arg s| \leqslant \tfrac{1}{2}\pi). \tag{4.11}$$

From (4.4), (4.6), and (4.10) it now follows that $K^+(-s) \to 1$ as $|s| \to \infty$ in the half-plane $|\arg s| \leqslant \frac{1}{2}\pi$ cut along $(1, \infty)$, and hence that, in the plane cut along $(-\infty, -1)$,

$$K^+(s) \to 1 \quad \text{as } |s| \to \infty, \tag{4.12}$$

uniformly in $\arg s$.

4.2. Now define

$$H(\mu) = \begin{Bmatrix} (1+\mu)K^+(\mu^{-1}) & (\text{case (i)}) \\ \dfrac{1+\mu}{1+k\mu}K^+(\mu^{-1}) & (\text{case (ii)}) \\ K^+(\mu^{-1}) & (\text{case (iii)}) \end{Bmatrix}. \tag{4.13}$$

Then $H(\mu)$ is regular and not zero in the μ-plane cut along $(-1, 0)$ except for a simple pole at infinity in case (i) and a simple pole at $\mu = -k^{-1}$ in case (ii).

By (4.13), (4.10), (4.6), and (4.3), it follows at once that $H(\mu)$ satisfies the functional relation (3.1).

If $\operatorname{re} s > 0$, we can take $b = 0$ in (4.8). Then, since $K(z)$ is an even function of z,

$$\ln K^+(s) = \frac{1}{2\pi i}\int\limits_0^{i\infty} \ln K(z)\left\{\frac{1}{z-s}-\frac{1}{z+s}\right\}dz = \frac{s}{\pi i}\int\limits_0^{i\infty}\frac{\ln K(z)}{z^2-s^2}\,dz.$$

On putting $s = \mu^{-1}$, $z = \zeta^{-1}$, we get

$$\ln K^+(\mu^{-1}) = \frac{\mu}{\pi i}\int\limits_{-i\infty}^{0}\frac{\ln K(\zeta^{-1})}{\zeta^2-\mu^2}\,d\zeta = \frac{\mu}{2\pi i}\int\limits_{-i\infty}^{i\infty}\frac{\ln K(\zeta^{-1})}{\zeta^2-\mu^2}\,d\zeta \quad (\operatorname{re}\mu > 0). \tag{4.14}$$

But [cf. (3) 115], if $\operatorname{re}\mu > 0$, $\operatorname{re}a > 0$,

$$\frac{\mu}{2\pi i}\int\limits_{-i\infty}^{i\infty}\frac{\ln(1-a^2\zeta^2)}{\zeta^2-\mu^2}\,d\zeta = -\ln(1+a\mu), \tag{4.15}$$

and so, on subtracting (4.15) with $a = 1$ from (4.14) and using (4.13) and (4.3), we get (4.1) in case (i). In case (ii) we have to subtract (4.15) with $a = 1$ and add (4.15) with $a = k$ to (4.14) to get (4.1). In case (iii), (4.14) is (4.1).

From (4.12) and (4.13) it follows that, in the μ-plane cut along $(-1, 0)$,

$$H(\mu) \to 1 \quad \text{as } |\mu| \to 0 \tag{4.16}$$

uniformly in $\arg\mu$.

4.3. We have now to prove that $H(\mu)$ satisfies (1.1). Let μ be any point of the μ-plane cut along $(-1, 0)$. Then $z = -\mu$ is any point of the z-plane cut along $(0, 1)$. From (4.13) and the fact that $K^+(0)$ is finite and not zero, it follows that

$$1/H(z) \to 0 \quad \text{as } z \to \infty \ \text{(case (i))},$$

$$1/H(z) \to \text{a finite limit} \quad \text{as } z \to \infty \ \text{(cases (ii), (iii))}.$$

Hence
$$\int\limits_{|z|=R}\frac{dz}{z(z+\mu)H(-z)} \to 0 \quad \text{as } R \to \infty. \tag{4.17}$$

Let Γ be a closed contour in the z-plane, cut along $(0, 1)$, which surrounds the cut but does not contain $z = -\mu$. Then, on using (4.17), we get

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{dz}{z(z+\mu)H(-z)} = \frac{1}{\mu H(\mu)}. \tag{4.18}$$

Let Γ consist of (i) γ_0 the circle $z = \epsilon e^{i\phi}$, where ϕ runs from 0 to 2π, (ii) λ the lower side of the cut from $x = \epsilon$ to $x = 1-\epsilon$, (iii) γ_1 the circle $z = 1+\epsilon e^{i\phi}$, where ϕ runs from $-\pi$ to π, (iv) λ' the upper side of the cut from $x = 1-\epsilon$ to $x = \epsilon$. For sufficiently small ϵ, Γ is in D and $-\mu$ is outside Γ. We shall let $\epsilon \to 0$. By (3.1), (4.18) can be written in the form

$$\frac{1}{H(\mu)} = \frac{\mu}{2\pi i}\int\limits_{\gamma_0}\frac{dz}{z(z+\mu)H(-z)} + \frac{\mu}{2\pi i}\int\limits_{\lambda\cup\gamma_1\cup\lambda'}\frac{H(z)T(z)}{z(z+\mu)}\,dz. \tag{4.19}$$

By (4.16), the contribution of γ_0 is $1+o(1)$; and by (2.5) that of γ_1 is $O(\epsilon\ln\epsilon) = o(1)$. Also, by (2.5) and (2.4),

$$T(z) = \begin{cases}\Omega(x)-i\pi x\Psi(x) & \text{on } \lambda,\\ \Omega(x)+i\pi x\Psi(x) & \text{on } \lambda',\end{cases}$$

where

$$\Omega(x) = 1-2x^2\int_0^1 \{\psi_1(y^2)-\psi_1(x^2)+y[\psi_2(y^2)-\psi_2(x^2)]\}\frac{dy}{x^2-y^2}-$$
$$-x\psi_1(x^2)\ln\frac{1+x}{1-x}-x^2\psi_2(x^2)\ln\frac{x^2}{1-x^2}. \quad (4.20)$$

Hence, when we let $\epsilon \to 0$, (4.19) becomes

$$\frac{1}{H(\mu)} = 1+\frac{\mu}{2\pi i}\int_0^1 \frac{H(x)}{x(x+\mu)}\{\Omega(x)-i\pi x\Psi(x)-\Omega(x)-i\pi x\Psi(x)\}\,dx,$$

i.e.
$$\frac{1}{H(\mu)} = 1-\mu\int_0^1 \frac{H(x)\Psi(x)}{x+\mu}\,dx,$$

and this is (1.1).

4.4. The following proof of the uniqueness is a little simpler than Crum's [see (**3**) 122], though the ideas used are the same.

Let $H_1(\mu)$ be any solution of (1.1) which is continuous in the interval $0 \leqslant \mu \leqslant 1$. By Theorem I, $1/H_1(\mu)$ is regular in the μ-plane cut along $(-1,0)$, it may have a simple zero at either $\mu = k^{-1}$ or at $\mu = -k^{-1}$ (case (ii)), and it satisfies (3.1). Hence

$$H_1(\mu)H_1(-\mu) = 1/T(\mu) = H(\mu)H(-\mu). \quad (4.21)$$

Let
$$\phi(\mu) = H(\mu)/H_1(\mu). \quad (4.22)$$

Then $\phi(\mu)$ is regular in the plane cut along $(-1,0)$ except, possibly, for a simple pole at infinity in case (i) or a simple pole at $\mu = -k^{-1}$ in case (ii). Since $H(\mu)$ is non-zero, the only possible zero of $\phi(\mu)$ is at $\mu = k^{-1}$ (case (ii)). From (4.21) we have

$$\phi(\mu)\phi(-\mu) = 1, \quad (4.23)$$

and it follows from this that $\phi(\mu)$ is regular in the whole μ-plane except, possibly, for a simple pole at infinity (case (i)), at $-k^{-1}$ (case (ii)), and an isolated singularity at $\mu = 0$. However, by letting $\mu \to \infty$ in (4.23), we see that

$$\phi(\mu) \to \pm 1 \quad \text{as } \mu \to \infty, \quad (4.24)$$

and $\phi(\mu)$ is regular there.

Consider $H_1(\mu)$ as $\mu \to 0$. If $\operatorname{re}\mu \geqslant 0$ and $0 \leqslant x \leqslant 1$, then $|\mu+x| \geqslant |\mu|$ and $|\mu+x| \geqslant x$. It follows from (1.1) that, if $0 < \eta < 1$,

$$\left|\frac{1}{H_1(\mu)}-1\right| \leqslant \int_0^\eta |H_1(x)|\Psi(x)\,dx+|\mu|\int_\eta^1 \frac{|H_1(x)|\Psi(x)}{x}\,dx$$
$$= o(1) \quad \text{as } |\mu| \to 0$$

by choosing first η and then μ. Hence

$$H_1(\mu) \to 1 \quad \text{as } |\mu| \to 0 \text{ for } \operatorname{re}\mu \geqslant 0. \tag{4.25}$$

By (4.16), the same is true of $H(\mu)$ and therefore of $\phi(\mu)$. From (4.23) it now follows that

$$\phi(\mu) \to 1 \quad \text{as } \mu \to 0 \tag{4.26}$$

in any manner. Hence $\phi(\mu)$ is regular at $\mu = 0$.

In cases (i) and (iii), $\phi(\mu)$ is a bounded integral function and it is therefore a constant, which must be unity. Thus the solution $H(\mu)$ is unique.

In case (ii), if $\phi(\mu)$ has a simple pole at $\mu = -k^{-1}$, it must also have a simple zero at $\mu = k^{-1}$ [by (4.23)]. The only function with this pole and zero, regular elsewhere, and satisfying (4.24) and (4.26) is

$$\phi(\mu) = \frac{1-k\mu}{1+k\mu}.$$

Hence the only other solution is

$$H_1(\mu) = \frac{1+k\mu}{1-k\mu} H(\mu).$$

The verification that this is also a solution is simple. See (**3**) 123.

Acknowledgement. I am indebted to the referee for pointing out that (4.12) is true uniformly in $\arg s$, thus effecting a great simplification in § 4.3, and for showing that $\Psi(z)$ need not be restricted to be an even function.

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A CONGRUENCE PROPERTY OF RAMANUJAN'S FUNCTION

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RAMANUJAN's function $\tau(n)$ is defined by

$$\sum_1^\infty \tau(n)x^n = x\prod_1^\infty (1-x^n)^{24} \quad (|x| < 1).$$

In (1), Bambah and Chowla state, without proof, the following congruence for $\tau(n)$:

$$\tau(n) \equiv 8n^4\sigma_3(n) - 14\Big\{2(1-n-n^3)\sigma_3(n) + \\ +(2n^2-3)\sigma(n) + \sum_{u+7v=n} \sigma(u)\sigma_3(7v)\Big\} \pmod{49},$$

with the added condition that $(n,7) = 1$. Here $\sigma_r(n)$ is the sum of the rth powers of all the divisors of n; $\sigma(n) = \sigma_1(n)$.

My purpose in this note is to obtain a second congruence for $\tau(n)$ for the modulus 49. This congruence will be simpler in form than the above congruence, will be very easy to obtain, and will hold even when 7 divides n.

Following Ramanujan [(2) 140], we set

$$Q = 1+240\sum_1^\infty \sigma_3(n)x^n,$$

$$R = 1-504\sum_1^\infty \sigma_5(n)x^n.$$

Then [(2) 144], $$1728\sum_1^\infty \tau(n)x^n = Q^3-R^2.$$

Now, from the relation 4 of Table I on page 141 of (2) we have

$$Q^3 \equiv Q\Big(1+39\sum_1^\infty \sigma_7(n)x^n\Big) \pmod{49},$$

and from relation 6 of that same table

$$5R^2 \equiv 5+7\sum_1^\infty \sigma_{11}(n)x^n \pmod{49}.$$

Hence

$$5.1728\sum_1^\infty \tau(n)x^n = 5(Q^3-R^2)$$

$$\equiv 5\Big(1+44\sum_1^\infty \sigma_3(n)x^n\Big)+$$

$$+5.39\sum_1^\infty \sigma_7(n)x^n\Big(1+44\sum_1^\infty \sigma_3(n)x^n\Big)-$$

$$-5-7\sum_1^\infty \sigma_{11}(n)x^n \pmod{49}.$$

If we now use the fact that $5.1728.46 \equiv 1 \pmod{49}$, reduce all coefficients *modulo* 49, and compare coefficients of x^n, we obtain

$$\tau(n) \equiv 26\sigma_3(n)+3\sigma_7(n)+21\sigma_{11}(n)+34U_{3,7}(n) \pmod{49},$$

where $$U_{r,s}(n) = \sum_1^{n-1} \sigma_r(k)\sigma_s(n-k) \quad (n>1),$$

$$U_{r,s}(1) = 0.$$

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A NOTE ON MATHIEU'S FUNCTIONS

By AUREL WINTNER (*Johns Hopkins University*)

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SOME time ago I initiated an existence theory of completely-monotone solutions for ordinary linear differential equations. The purpose of this note is to point out a relation which the general theory supplies for the particular case of Mathieu's equation. The explicit form of the 'density function' in the resulting integral representation does not seem to follow from classical formulae. On the other hand, that result can hardly be of a type which is not fundamentally 'explicit' in nature; in fact, the device to be applied fails in every case distinct from Mathieu's choice of the coefficient function.

It is known† that, if both coefficient functions of a differential equation

$$d^2y/dx^2+a_1(x)\,dy/dx-a_2(x)y = 0, \tag{1}$$

where

$$0 < x < \infty, \tag{2}$$

have derivatives of arbitrarily high order which, at every point of the half-line (2), satisfy the inequalities

$$(-d/dx)^m a_j(x) \geqslant 0$$

for $m = 0, 1,\ldots$, then (1) possesses on (2) some solution $y = y(x) \neq 0$ which is representable in the form

$$y(x) = \int\limits_0^\infty e^{-xt}\,dF(t), \tag{3}$$

where $F(t)$ is a certain function satisfying

$$0 \not\equiv dF(t) \geqslant 0 \quad \text{for } 0 \leqslant t < \infty \tag{4}$$

and rendering the integral (3) convergent at every point of (2).

Clearly, the conditions imposed on the functions $a_j(x)$ are satisfied if (though not only if)

$$a_j(x) = \sum_{n=0}^\infty a_{j,n}/x^n \quad (a_{j,n} \geqslant 0), \tag{5}$$

provided that both Taylor series (5) (in $1/x$) converge on the whole of (2); in other words, if $a_1(1/z)$ and $a_2(1/z)$ are entire functions of z and have at $z = 0$ real, non-negative derivatives only (this includes the 0th derivatives).

† Cf. G. Doetsch, *Handbuch der Laplace-Transformation*, 2, part 1 (1955), 399–404.

Quart. J. Math. Oxford (2), 8 (1957), 143-5.

It does not seem to have been observed thus far that the case (5) of the general theorem has a curious consequence for Mathieu's equation

$$d^2y/d\phi^2+(p-2q\cos 2\phi)y = 0 \tag{6}$$

if both constants p, q are real and non-negative (and the angular variable ϕ is real). Actually, a connexion between (6) and a particular case of the case (5) of (1) can be obtained by an adaptation (to the real case) of the standard substitution† which 'rationalizes' (6).

First, (6) is identical with

$$d^2y/ds^2-(p-2q\cosh 2s)y = 0 \tag{7}$$

if $\phi = is$. Next, if $q > 0$, then the substitution $q^{\frac{1}{2}}e^{-s} = x$ transforms the line $-\infty < s < \infty$ and the equation (7) into (2) and

$$xd(x\,dy/dx)/dx-(p+q/x^2+x^2/q)y = 0, \tag{8}$$

respectively, since $ds = (-dx)/x$ and $e^{2s} = x^2/q$. But (8) can be written in the form (1) by placing

$$a_1(x) = 1/x, \qquad a_2(x) = 1/q+p/x^2+q/x^4, \tag{9}$$

and the choice (9) of the functions (5) complies with all inequalities of (5) if $q > 0$ and $p \geqslant 0$.

Accordingly, it follows from the general theorem on (1) that (8) must possess on the half-line (2) some solution $y = y(x) \neq 0$ which, on (2), is representable as a (convergent) integral (3) in which $F(t) = F_{pq}(t)$ is a certain function satisfying (4). But the convergence of (3) for $x > 0$ implies the convergence of (3) for $\operatorname{re} x > 0$. On the other hand, (8) is identical with (6) by virtue of $x = q^{\frac{1}{2}}e^{i\phi}$ and, since $x = q^{\frac{1}{2}}\cos\phi$ for real ϕ, the proviso $\operatorname{re} x > 0$ is satisfied if $q^{\frac{1}{2}} > 0$ and

$$-\tfrac{1}{2}\pi < \phi < \tfrac{1}{2}\pi. \tag{10}$$

Since the coefficient function of (6), and hence, (6) being linear, every solution y of (6) as well, is an entire function of ϕ if, for a moment, ϕ is allowed to be complex, it follows that (6) has a solution $y = y(\phi) \not\equiv 0$ which, on the open interval (10), is representable as an absolutely convergent integral

$$y(\phi) = \int\limits_0^\infty \exp(-q^{\frac{1}{2}}e^{i\phi}t)\,dF(t), \tag{11}$$

where $F = F_{pq}$ is the same function of t ($\geqslant 0$) as in (3).

Let the integration variable t be replaced by $q^{-\frac{1}{2}}t$, where $q^{\frac{1}{2}}$ (> 0) is

† Cf., e.g., G. Doetsch, op. cit. 397.

fixed. Then the half-line $0 \leqslant t < \infty$ goes over into itself, and the real part of (11) becomes

$$y(\phi) = \int_0^\infty \exp(-t\cos\phi)\cos(t\sin\phi)\, dF(t). \tag{12}$$

Since (6) is a real equation, it follows that (6) has a solution which is representable as an absolutely convergent integral (12) on (10). Needless to say, the imaginary part of (11) is another solution: that is, (12) remains a solution if its second 'cos' is changed to 'sin'. But it is clear that, if $y(\phi)$ denotes this sine integral, then both $y(\phi)$ and $dy(\phi)/d\phi$ become 0 at $\phi = 0$, which means that the sine integral is the trivial solution, $y(\phi) \equiv 0$, of (6). Such is not the case for the cosine integral since (12) and (4) show that $y(\phi) > 0$ at $\phi = 0$.

Note that the length of the interval (10), the interval on which (12) is valid, is precisely the period π of the coefficient function of (6). This point is essential since, except in the case $p^* = p^*(q)$ of an eigenvalue $p = p^*$, the classical theory (Fuchs–Floquet) supplies for (6) the existence of two linearly independent solutions, say $y = y_1(\phi)$ and $y = y_2(\phi)$, both of which are of the form $e^{i\lambda t}f(t)$, where the two 'characteristic exponents' $\lambda = \lambda(p,q)$ need not be real and both functions $f(t)$ are of period π (in Ince's 'logarithmic' case of an eigenvalue $p = p^*$, only $y_1(t)$ is of the form $e^{i\lambda t}f(t)$, where $f(t)$ is of period π.)†

This raises the question of the explicit determination of the function (4) occurring in (12) (when $p \geqslant 0$ and $q > 0$ are arbitrary). An answer could perhaps be computed from a known integral connexion‡ between cylinder functions and the solutions of (6).§ In the *particular case of eigenfunctions*, that is, of Mathieu functions in the customary sense of the term, the hyperbolic form of Whittaker's integral equations|| seems to be the appropriate point of departure for such a verification.

† Cf., e.g., E. G. C. Poole, *Introduction to the theory of linear differential equations* (Oxford, 1936), 171–2 and 179–82.

‡ Cf. G. Doetsch, op. cit. 398–9.

§ Concerning a corresponding question arising for Whittaker's normal form of the confluent hypergeometric equation (an equation which, in view of (9), can be thought of as a limiting case of Mathieu's equation), cf. sections 11–12, formulae (36)–(43), of a paper to appear in the *Rendiconti del Circolo Matematico di Palermo*.

|| Cf. E. G. C. Poole, op. cit. 197–8, formulae (iv).

THE STURM–LIOUVILLE PROBLEM FOR FOURTH-ORDER DIFFERENTIAL EQUATIONS

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1. In this paper I consider the direct extension to fourth-order ordinary differential equations of the analysis of the Sturm–Liouville problem given in his book (**1**) by E. C. Titchmarsh. This type of problem has been considered, for general-order equations, by many writers [see in particular Birkhoff (**2**) and Tamarkin (**3**)]. Here we consider some of the explicit constructions for the eigenfunctions; the fact that equations of the fourth and higher order are occurring in mathematical physics suggests that such information might be useful [see Eringen (**5**) and Chandrasekhar (**4**)].

The method stems from a new representation of the Green's function for a boundary-value problem. The method can be at once extended to equations of higher order and to more general boundary conditions than I consider here. Most of the information presented is taken from a D.Phil. thesis (May, 1955) accepted by the University of Oxford. In some places excessive details have been omitted but the results obtained are stated in full.

The work is based on the book Titchmarsh (**1**) and some ideas from the paper by Kodaira (**6**).

The author wishes to express his gratitude to Professor E. C. Titchmarsh.

2. The boundary-value problem (B.V.P.) we consider is the differential equation

$$Ly \equiv y^{(4)}-[q_1(x)y^{(1)}]^{(1)}+q_0(x)y = \lambda y \quad (a \leqslant x \leqslant b), \tag{2.1}$$

where $y^{(r)}$ denotes the differential coefficient of order r with respect to x, with the Sturmian boundary conditions

$$U_i(a,y) \equiv \sum_{j=0}^{3} (-1)^{j+1}\alpha_i^{4-j}y^{(j)}(a) = 0 \quad (i = 1, 2), \tag{2.2}$$

$$U_i(b,y) \equiv \sum_{j=0}^{3} (-1)^{j+1}\beta_i^{4-j}y^{(j)}(b) = 0 \quad (i = 1, 2). \tag{2.3}$$

The unusual form of the α_i^j and β_i^j in the boundary conditions is chosen

for convenience in dealing with the problem of asymptotic expansions of certain solutions of (2.1) which may perhaps be discussed in a later note.

We require the following conditions:

(i) the closed interval $[a, b]$ is finite;
(ii) $q_0(x)$, $q_1(x)$, and $q_1^{(1)}(x)$ are real-valued and continuous in $[a, b]$;
(iii) λ is the complex eigenvalue parameter ($\lambda = u+iv$);
(iv) all the α_i^j and β_i^j are real-valued;
(v) the row vectors (α_1^j) and (α_2^j) ($j = 1,\ldots, 4$) are linearly independent and satisfy

$$q_1(a)\{\alpha_1^2\,\alpha_2^1-\alpha_1^1\,\alpha_2^2\}+\{\alpha_1^4\,\alpha_2^1-\alpha_2^4\,\alpha_1^1+\alpha_1^2\,\alpha_2^3-\alpha_1^3\,\alpha_2^2\} = 0; \tag{2.4}$$

(vi) the β_i^j satisfy conditions similar to (v) with a, α replaced by b, β.

It is easily seen that the operator L of (2.1) is self-adjoint [see Ince (**7**) 125 and Kamke (**9**) 76]. The condition (2.4), together with the similar condition for the β_i^j implies that the B.V.P. (2.1) to (2.3) is self-adjoint [see Birkhoff (**2**) and Kamke (**9**) 187]. This follows from results in the paper by Latshaw (**10**) on the algebraic properties of differential systems [see (**10**) 97].

Any value of λ for which the B.V.P. has a non-trivial solution is called an 'eigenvalue' and the solution an 'eigenfunction'. For any eigenvalue λ the number of eigenfunctions linearly independent over $[a, b]$ is called the 'index' of the eigenvalue and denoted by $k(\lambda)$.

3. As in Titchmarsh (**1**) 62 and Ince (**7**) 270, the B.V.P. of § 2 is a canonical form of a more general class of problems in which the operator L takes the more general self-adjoint form

$$Ly \equiv [f_2(x)\,y^{(2)}]^{(2)}+[f_1(x)\,y^{(1)}]^{(1)}+f_0(x)\,y = \lambda g(x)\,y. \tag{3.1}$$

A self-adjoint B.V.P. composed of (3.1) and Sturmian boundary conditions can, in certain circumstances, be reduced to the type of problem given in § 2. Several writers have discussed this problem, in particular Sternberg (**12**) and Davis (**11**). I only note here, since it does not appear in the books or papers, that the transformation required to transform (3.1) into (2.1) is

$$\xi = \int^{x} \left\{\frac{g(t)}{f_2(t)}\right\}^{\frac{1}{4}} dt, \qquad \eta(\xi) = [\{g(x)\}^3 f_2(x)]^{\frac{1}{8}}. \tag{3.2}$$

4. Let $\{\alpha_r;\ 0 \leqslant r \leqslant 3\}$ be any set of constant complex numbers, not all zero. We shall use the functional notation $\phi(\xi \mid x, \lambda)$ to represent that

solution of the differential equation (2.1), with independent variable x, which satisfies the following initial conditions at $x = \xi$, where $\xi \in [a, b]$,

$$[\phi^{(r)}(\xi \mid x, \lambda)]_{x=\xi} = \alpha_r \quad (0 \leqslant r \leqslant 3). \tag{4.1}$$

The conditions laid down in § 2 and the well-known existence theorems for linear differential equations [see for example Kamke (**8**)] tell us that $\phi(\xi \mid x, \lambda)$ exists uniquely and is not identically zero. Also $\phi(\xi \mid x, \lambda)$ and its first four derivatives with respect to x are integral functions of the complex variable λ.

If α_r $(0 \leqslant r \leqslant 3)$ are all real, then $\phi(\xi \mid x, \lambda)$ is real-valued when λ is real, and consequently, by the principle of reflection,

$$\phi(\xi \mid x, \bar{\lambda}) = \bar{\phi}(\xi \mid x, \lambda) \quad \text{for all } \lambda. \tag{4.2}$$

I use the notation $\{\eta_i(\xi \mid x, \lambda);\ 1 \leqslant i \leqslant 4\}$ for the 'unit set' of (2.1) at $x = \xi$, i.e. that fundamental set which satisfies

$$[\eta_i^{(j-1)}(\xi \mid x, \lambda)]_{x=\xi} = \delta_{ij} \quad (1 \leqslant i, j \leqslant 4), \tag{4.3}$$

where δ_{ij} is the Kronecker delta.

5. As with all B.V.P. of the type in § 2, properties of the eigenfunctions stem from the Green's formula. For the operator L given in (2.1) this can be written as [see Ince (**7**) 211]

$$\int_{x_1}^{x_2} \{v\,Lu - u\,Lv\}\,dx = [P_x(u\,v)]_{x_1}^{x_2} \quad (a \leqslant x_1 < x_2 \leqslant b), \tag{5.1}$$

where $u(x)$ and $v(x)$ are any two functions possessing suitable derivatives and $P_x(u\,v)$ is the so-called 'bilinear concomitant' of L. Explicitly

$$P_x(u\,v) = q_1(x)\,[u(x)\,v^{(1)}(x) - u^{(1)}(x)\,v(x)] +$$
$$+ [u^{(3)}(x)\,v(x) - u^{(2)}(x)\,v^{(1)}(x) + u^{(1)}(x)\,v^{(2)}(x) - u(x)\,v^{(3)}(x)]. \tag{5.2}$$

The implication of the self-adjoint nature of our boundary-value problem [see (2.4) and the remarks after that condition] is that, if $u(x, \lambda_1)$ and $v(x, \lambda_2)$ are eigenfunctions for λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$, then

$$P_a\{u(x, \lambda_1)\,v(x, \lambda_2)\} = P_b\{u(x, \lambda_1)\,v(x, \lambda_2)\}. \tag{5.3}$$

This result follows from the definition of the adjoint system [see Birkhoff (**2**), Latshaw (**10**), and Kamke (**9**) 187].

It is clear that, if $u(x, \lambda_1)$ and $v(x, \lambda_2)$ satisfy the above conditions then (5.1) and (5.3) give, since $\lambda_1 \neq \lambda_2$,

$$\int_a^b u(x, \lambda_1)\,v(x, \lambda_2)\,dx = 0, \tag{5.4}$$

which expresses the usual orthogonal property of eigenfunctions for *real-valued* self-adjoint systems.

Another important property of $P_x(u\,v)$ is that, if $u(x,\lambda)$ and $v(x,\lambda)$ both satisfy $Ly = \lambda y$ for the *same* value of λ, then $P\{u(x,\lambda)\,v(x,\lambda)\}$ is independent of x and depends only on λ. This follows at once from the Green's formula (5.1).

6. In this section I exploit an idea due to Kodaira [(**6**) 521]. Consider one of the boundary conditions $U_i(a,y)$ given in (2.2). Let us for the moment write this as

$$\alpha\, y(a)+\beta\, y^{(1)}(a)+\gamma\, y^{(2)}(a)+\delta\, y^{(3)}(a) = 0. \tag{6.1}$$

We seek to represent this boundary condition in the form

$$P_a\{y(x)\,\phi(x)\} = 0, \tag{6.2}$$

where $\phi(x)$ is, as yet, some undefined function of x with a suitable number of derivatives. If (6.1) and (6.2) are to be equivalent, then expanding (6.2) and comparing coefficients, we obtain

$$\begin{aligned} \phi(a) &= \delta, & \phi^{(1)}(a) &= -\gamma, \\ \phi^{(2)}(a) &= \beta+q_1(a)\,\delta, & \phi^{(3)}(a) &= -\alpha-q_1(a)\,\gamma. \end{aligned} \tag{6.3}$$

To determine $\phi(x)$ completely we now demand that it satisfy the differential equation $Ly = \lambda y$ of § 2. The existence theorem in § 4 tells us that a function $\phi(a \mid x,\lambda)$ exists uniquely which satisfies (2.1) and the initial conditions (6.3) at $x = a$. Thus every boundary condition of the type (2.2), and similarly for (2.3), can be recast in the form (6.2) where ϕ is uniquely determined.

To test whether any solution $y(x,\lambda)$ of $Ly = \lambda y$ satisfies a condition such as (6.1) we consider

$$P\{y(x,\lambda)\,\phi(a \mid x,\lambda)\}, \tag{6.4}$$

where any value of x may be used since from the last paragraph of § 5 this expression is independent of x. If (6.4) is zero, then (6.1) is satisfied by $y(x,\lambda)$.

This shows that there exist four functions

$$\phi_i(a \mid x,\lambda) \quad (i = 1, 2), \qquad \chi_i(b \mid x,\lambda) \quad (i = 1, 2) \tag{6.5}$$

all solutions of $Ly = \lambda y$, associated respectively with the boundary conditions

$$U_i(a,y) = 0 \quad (i = 1, 2), \qquad U_i(b,y) = 0 \quad (i = 1, 2).$$

These functions ϕ_i and χ_i are the natural extension of ϕ and χ in Titchmarsh (**1**), Chapter I.

After taking account of the difference in notation between (2.2) and (6.1) we see that the functions $\phi_i(a \mid x, \lambda)$ satisfy $Ly = \lambda y$ and the following initial conditions

$$[\phi_i(a \mid x,\lambda)]_{x=a} = \alpha_i^1, \qquad [\phi_i^{(1)}(a \mid x,\lambda)]_{x=a} = \alpha_i^2,$$
$$[\phi_i^{(2)}(a \mid x,\lambda)]_{x=a} = \alpha_i^3 + q_1(a)\,\alpha_i^1,$$
$$[\phi_i^{(3)}(a \mid x,\lambda)]_{x=a} = \alpha_i^4 + q_1(a)\,\alpha_i^2. \tag{6.6}$$

The functions $\chi_i(b \mid x, \lambda)$ satisfy similar conditions at $x = b$ in terms of the β_i^j.

The importance of these 'boundary-condition' functions ϕ_i and χ_i is that we can manipulate the boundary-value problem of § 2 without having to refer explicitly to the constants α_i^j and β_i^j.

Since we stipulated in § 2 that the sets α_1^j and α_2^j are to be linearly independent, it follows that ϕ_1 and ϕ_2 are linearly independent over $[a, b]$ for all λ. Likewise for χ_1 and χ_2.

Also, since U_1 and U_2 of (2.2) are homogeneous, it is clear that ϕ_1 and ϕ_2 are unique only up to linear combination: that is, a non-singular linear transformation of ϕ_1 and ϕ_2 into a new pair does not alter the boundary-value problem.

7. The above method of representing the boundary conditions leads to the following method of representing the self-adjointness of our B.V.P. If we calculate the value of $P(\phi_1 \phi_2)$, which is soon seen to be independent of x *and* λ, we find that it is identical with the left-hand side of (2.4). Similarly for $P(\chi_1 \chi_2)$. Thus the self-adjointness of our problem can be expressed as

$$P(\phi_1 \phi_2) = 0, \qquad P(\chi_1 \chi_2) = 0. \tag{7.1}$$

This last result is more convenient to use than the Latshaw condition (2.4). See also Kodaira [(**6**) 521].

I introduce the notation

$$P_{ij} \equiv P_{ij}(\lambda) \equiv P\{\phi_i(a \mid x,\lambda)\chi_j(b \mid x,\lambda)\} \quad (1 \leqslant i,\ j \leqslant 2), \tag{7.2}$$

valid by virtue of the last paragraph of § 5.

8. Let $\{y_i(x,\lambda);\ 1 \leqslant i \leqslant r\}$ be any r $(1 \leqslant r \leqslant 4)$ solutions of $Ly = \lambda y$. I shall use

$$W_x(y_1 y_2 \ldots y_r)(\lambda) \equiv W\{y_1(x,\lambda)\, y_2(x,\lambda) \ldots y_r(x,\lambda)\} \tag{8.1}$$

to represent the Wronskian of order r of this set of functions [see Ince (**7**) 116–21 for the definition and properties of W].

In the particular case $r = 4$ we have the following expression for W

in terms of the P function: this expression is obtained from the paper by Kodaira [(**6**) 504].

$$\begin{aligned}W\{y_1(x,\lambda)\ldots y_4(x,\lambda)\} = &-P\{y_1(x,\lambda)\,y_2(x,\lambda)\}\,P\{y_3(x,\lambda)\,y_4(x,\lambda)\}+\\ &+P\{y_1(x,\lambda)\,y_3(x,\lambda)\}\,P\{y_2(x,\lambda)\,y_4(x,\lambda)\}-\\ &-P\{y_1(x,\lambda)y_4(x,\lambda)\}\,P\{y_2(x,\lambda)\,y_3(x,\lambda)\}.\end{aligned} \tag{8.2}$$

This identity shows that $W_x(y_1\ldots y_4)(\lambda)$, of *any* four solutions, is in fact independent of x and depends only on λ. [See the end of § 5.] This follows also from the fact that the coefficient of $y^{(3)}$ in (2.1) is identically zero [see Ince (**7**) 119]. Compare Titchmarsh (**1**) Chapter I.

A particular application of (8.2) to the four functions ϕ_i and χ_i of (6.5) gives, with (7.1) and (7.2), the definition

$$W(\lambda) \equiv W_x(\phi_1\,\phi_2\,\chi_1\,\chi_2)(\lambda) = P_{11}(\lambda)\,P_{22}(\lambda)-P_{12}(\lambda)\,P_{21}(\lambda). \tag{8.3}$$

Clearly $W(\lambda)$ and the $P_{ij}(\lambda)$ are integral functions of λ.

9. As in Titchmarsh [(**1**) 12] we can prove that all the eigenvalues of the B.V.P. (2.1) to (2.3) are *real*, when we use (4.2) and (5.4). This also implies that, since L is real-valued, we can take, without loss of generality, all *eigenfunctions* to be *real-valued*.

In Titchmarsh [(**1**) Chapter I] the eigenvalues are given by the roots of the transcendental equation

$$W\{\phi(x,\lambda),\,\chi(x,\lambda)\} = \omega(\lambda) = 0. \tag{9.1}$$

The extension of this is

THEOREM 9.1. *A necessary and sufficient condition that λ should be an eigenvalue is that it be a root of the equation*

$$W(\lambda) = 0, \tag{9.2}$$

where $W(\lambda)$ is defined by (8.3).

Proof. Suppose that for some value of λ, say $\lambda = \mu$,

$$W(\mu) \neq 0. \tag{9.3}$$

Then, from Ince (**7**) 119, the four functions $\phi_i(x,\mu)$ and $\chi_i(x,\mu)$ $(i = 1, 2)$ form a fundamental set for $Ly = \mu y$. Thus, if $y(x,\mu)$ is *any* non-trivial solution of this equation

$$y(x,\mu) = \sum_{i=1}^{2}\{\alpha_i\,\phi_i(x,\mu)+\beta_i\,\chi_i(x,\mu)\},$$

where not *all* the α_i and β_i are zero.

If $y(x,\mu)$ is to satisfy the boundary conditions (2.2) and (2.3), then, using $P\{\dots\phi_i(x,\mu)\}$, etc., as a linear operator and from (6.2), we have

$$\sum_{i=1}^{2}\{\alpha_i P(\phi_i\phi_j)(\mu)+\beta_i P(\chi_i\phi_j)(\mu)\} = 0 \quad (j = 1, 2),$$

$$\sum_{i=1}^{2}\{\alpha_i P(\phi_i\chi_j)(\mu)+\beta_i P(\chi_i\chi_j)(\mu)\} = 0 \quad (j = 1, 2).$$

For these linear homogeneous equations to have a non-trivial solution it is necessary that the determinant of the coefficients, say D, should be zero. However, it is readily calculated that

$$D = -\{W(\mu)\}^2 \neq 0, \quad \text{by (9.3).}$$

Thus $\lambda = \mu$ is not an eigenvalue.

Suppose on the other hand that, for $\lambda = \mu$,

$$W(\mu) = 0. \tag{9.4}$$

Then the Wronskian of $\phi_i(x,\mu)$ and $\chi_i(x,\mu)$ $(i = 1, 2)$ is identically zero for $x \in [a,b]$, and we have a linear relationship

$$\sum_{i=1}^{2}\{\alpha_i\phi_i(x,\mu)-\beta_i\chi_i(x,\mu)\} \equiv 0, \tag{9.5}$$

where not *both* of α_1 and α_2 are zero and not *both* of β_1 and β_2 are zero from the last paragraph but one of § 5.

Consider then the function $y(x,\mu)$ defined by

$$y(x,\mu) \equiv \sum_{i=1}^{2}\alpha_i\phi_i(x,\mu).$$

Clearly $y(x,\mu)$ is a non-trivial solution of (2.1) and satisfies (2.2) from the definition. The identity (9.5) implies that it also satisfies (2.3). Then $y(x,\mu)$ is an eigenfunction and μ is an eigenvalue. This completes the proof.

We note that, since all eigenvalues are real, all the roots of (9.2) are real. In § 17 I have something to say of the order of the zeros of $W(\lambda)$.

10. The ideas of §§ 6, 7 lead to a new expression for the Green's function of the B.V.P. given in § 2. The general theory for the Green's function of a homogeneous B.V.P. will be found in Ince (**7**), Kamke (**9**), and Courant and Hilbert (**13**). I adopt here the definition and sign-convention given in (**13**) 362 (also 353).

As with all the problems of the type in § 2 we can construct the Green's function $G(x,\xi;\lambda)$ only in the situation when

$$W(\lambda) \neq 0, \tag{10.1}$$

i.e. when λ is *not* an eigenvalue. Clearly with this restriction the four

functions $\phi_i(a \mid x, \lambda)$ and $\chi_i(b \mid x, \lambda)$ form a fundamental set of (2.1), and we can write [compare Ince (7) 254], for $\xi \in (a, b)$,

$$G(x, \xi; \lambda) = \begin{cases} \sum_{i=1}^{2} \{\alpha_i \phi_i(a \mid x, \lambda) + A_i \chi_i(b \mid x, \lambda\} & (a \leqslant x < \xi), \\ \sum_{i=1}^{2} \{\beta_i \chi_i(b \mid x, \lambda) + B_i \phi_i(a \mid x, \lambda)\} & (\xi < x \leqslant b), \end{cases} \tag{10.2}$$

where the α_i, etc., yet to be determined, will depend in general on ξ and λ.

Since G has to satisfy the boundary conditions (2.2), we have, as in § 9, for $i = 1, 2$,

$$P\{G(x, \xi; \lambda), \phi_i(a \mid x, \lambda)\} = 0$$
$$= -\{A_1 P_{i1}(\lambda) + A_2 P_{i2}(\lambda)\}. \tag{10.3}$$

Since the coefficients of the linear homogeneous equations (10.3) for A_1 and A_2 have, by (8.3), the determinant

$$P_{11}(\lambda) P_{22}(\lambda) - P_{12}(\lambda) P_{21}(\lambda) = W(\lambda) \neq 0, \quad \text{by (10.1)},$$

it is clear that $$A_1 = A_2 = 0.$$

Similarly $$B_1 = B_2 = 0.$$

With this done *any* choice of α_1, α_2 and β_1, β_2 implies that $G(x, \xi; \lambda)$ satisfies (2.2) and (2.3). To determine the α_i and β_i we use the discontinuity in the third derivative of G at $x = \xi$ [see Courant and Hilbert (13) 362] to obtain, for $1 \leqslant j \leqslant 4$,

$$\sum_{i=1}^{2} \{\alpha_i \phi_i^{(j-1)}(a \mid \xi, \lambda) - \beta_i \chi_i^{(j-1)}(b \mid \xi, \lambda)\} = \delta_{j4}, \tag{10.4}$$

where δ_{ij} is the Kronecker delta. The determinant of the coefficients of this set of non-homogeneous linear equations is, on inspection, $W(\lambda)$, which by (10.1) is not zero. Thus (10.4) serve to determine the α_i and β_i uniquely.

Explicitly we obtain, by the usual algebraic processes,

$$\left.\begin{aligned} \alpha_1 &= \frac{W_\xi(\chi_1 \phi_2 \chi_2)(\lambda)}{W(\lambda)}, & \alpha_2 &= \frac{W_\xi(\phi_1 \chi_1 \chi_2)(\lambda)}{W(\lambda)} \\ \beta_1 &= \frac{W_\xi(\phi_1 \phi_2 \chi_2)(\lambda)}{W(\lambda)}, & \beta_2 &= \frac{W_\xi(\phi_1 \chi_1 \phi_2)(\lambda)}{W(\lambda)} \end{aligned}\right\}, \tag{10.5}$$

where the notation of (8.1) is employed.

This unique determination of α_i and β_i proves the unique existence of $G(x, \xi; \lambda)$ under the restriction (10.1).

11. We can put the above determination of $G(x,\xi;\lambda)$ in a more explicit form by using the unit set defined in (4.3). It is readily seen that we can write

$$W_\xi(\chi_1\,\phi_2\,\chi_2)(\lambda) = W\{\chi_1(x,\lambda)\,\phi_2(x,\lambda)\,\chi_2(x,\lambda)\,\eta_4(\xi\mid x,\lambda)\} \tag{11.1}$$

for any value of x in $[a,b]$. We now expand (11.1) by use of (8.2) to obtain

$$W_\xi(\chi_1\,\phi_2\,\chi_2)(\lambda) = -P(\chi_1\,\phi_2)\,P\{\chi_2(x,\lambda)\,\eta_4(\xi\mid x,\lambda)\}- \\ -P\{\chi_1(x,\lambda)\,\eta_4(\xi\mid x,\lambda)\}\,P(\phi_2\,\chi_2).$$

By explicit expansion of P from (5.2) we have also

$$P\{\chi_i(x,\lambda)\,\eta_4(\xi\mid x,\lambda)\} = -\chi_i(\xi,\lambda) \quad (i=1,2),$$

and thus

$$W_\xi(\chi_1\,\phi_2\,\chi_2)(\lambda) = P_{22}(\lambda)\,\chi_1(\xi,\lambda)-P_{21}(\lambda)\,\chi_2(\xi,\lambda), \tag{11.2}$$

which is in fact valid for all λ and ξ in $[a,b]$.

In a similar manner

$$W_\xi(\phi_1\,\chi_1\,\chi_2)(\lambda) = P_{11}(\lambda)\,\chi_2(\xi,\lambda)-P_{12}(\lambda)\,\chi_1(\xi,\lambda), \tag{11.3}$$

$$W_\xi(\phi_1\,\phi_2\,\chi_2)(\lambda) = P_{22}(\lambda)\,\phi_1(\xi,\lambda)-P_{12}(\lambda)\,\phi_2(\xi,\lambda), \tag{11.4}$$

$$W_\xi(\phi_1\,\chi_1\,\phi_2)(\lambda) = P_{11}(\lambda)\,\phi_2(\xi,\lambda)-P_{21}(\lambda)\,\phi_1(\xi,\lambda). \tag{11.5}$$

If we substitute these expressions in the original form (10.2) of G, we obtain

$$G(x,\xi;\lambda) = \frac{P_{22}(\lambda)\,\chi_1(\xi,\lambda)-P_{21}(\lambda)\,\chi_2(\xi,\lambda)}{W(\lambda)}\,\phi_1(x,\lambda)+ \\ +\frac{P_{11}(\lambda)\,\chi_2(\xi,\lambda)-P_{12}(\lambda)\,\chi_1(\xi,\lambda)}{W(\lambda)}\,\phi_2(x,\lambda) \quad \text{for } a\leqslant x<\xi;$$

and

$$= \frac{P_{22}(\lambda)\,\phi_1(\xi,\lambda)-P_{12}(\lambda)\,\phi_2(\xi,\lambda)}{W(\lambda)}\,\chi_1(x,\lambda)+ \\ +\frac{P_{11}(\lambda)\,\phi_2(\xi,\lambda)-P_{21}(\lambda)\,\phi_1(\xi,\lambda)}{W(\lambda)}\,\chi_2(x,\lambda) \quad \text{for } \xi<x\leqslant b. \tag{11.6}$$

This is our required expression for G and is the natural extension of the representation of the Green's function used in Titchmarsh (**1**) Chapter I.

We note that (11.6) brings out the symmetrical property of $G(x,\xi;\lambda)$ in x and ξ: this is a necessary conclusion for self-adjoint B.V.P. [see Ince (**7**) 256].

We remark also that by the type of argument used to prove (11.2) any Wronskian of order three of (2.1) is also a solution of (2.1)—a general result which has been proved for self-adjoint operators.

12. As in Titchmarsh [(**1**) 7] we define a function $\Phi(x,\lambda)$ by

$$\Phi(x,\lambda) \equiv \int_a^b G(x,\xi;\lambda) f(\xi)\, d\xi, \tag{12.1}$$

for x in $[a,b]$ and λ not equal to any eigenvalue. Here $f(x)$ is any function which belongs to the class $L^p[a,b]$ for some $p \geqslant 1$. If also $f(x)$ is continuous in $[a,b]$, then it can be shown that Φ satisfies the non-homogeneous equation

$$L\,\Phi(x,\lambda) = \lambda\,\Phi(x,\lambda) - f(x).$$

To obtain an eigenfunction expansion of the arbitrary function $f(x)$ we have to integrate $\Phi(x,\lambda)$ round a large contour in the λ-plane [see Titchmarsh (**1**)]. This process demands a knowledge of the residue of Φ at its poles, i.e. at the zeros of $W(\lambda)$.

13. For any value of λ let $r(\lambda)$ denote the rank of the 2×2 matrix $[P_{ij}(\lambda)]$. With this and the definition of the index $k(\lambda)$ given in § 2 we have

THEOREM (13.1). *For all values of* λ

$$r(\lambda)+k(\lambda) = 2. \tag{13.1}$$

Proof. When $W(\lambda) \neq 0$, then, from (8.3), $r(\lambda) = 2$, and from Theorem (9.1), $k(\lambda) = 0$; then (13.1) follows in this case.

Suppose then that $W(\lambda) = 0$, i.e. that λ is an eigenvalue. Then we have at least one eigenfunction $y(x,\lambda)$. Consider the third-order Wronskian $W_x(y\,\phi_1\,\phi_2)(\lambda)$; by the method used in § 11 we can write, for any t in $[a,b]$,

$$\begin{aligned} W_x(y\,\phi_1\,\phi_2)(\lambda) &= W\{y(t,\lambda)\,\phi_1(t,\lambda)\,\phi_2(t,\lambda)\,\eta_4(x\mid t,\lambda)\} \\ &= 0 \quad (x\in[a,b]), \end{aligned}$$

from (8.2) since, by hypothesis, $y(x,\lambda)$ satisfies the boundary conditions at $x = a$. Thus we have a linear relationship

$$y(x,\lambda) = \alpha_1\,\phi_1(x,\lambda)+\alpha_2\,\phi_2(x,\lambda) \quad (x\in[a,b]).$$

Since $y(x,\lambda)$ also satisfies the boundary conditions at $x = b$,

$$\begin{aligned} P(y\,\chi_1) &= \alpha_1\,P_{11}(\lambda)+\alpha_2\,P_{21}(\lambda) = 0, \\ P(y\,\chi_2) &= \alpha_1\,P_{12}(\lambda)+\alpha_2\,P_{22}(\lambda) = 0, \end{aligned}$$

which must be satisfied by α_1 and α_2. Now the number of distinct solutions of these homogeneous equations is given by the rank $r(\lambda)$ of $[P_{ij}(\lambda)]$.

If $r(\lambda) = 1$, then there is a unique ratio for α_1 and α_2, and $k(\lambda) = 1$.

If $r(\lambda) = 0$, then there are two independent ratios for α_1 and α_2, and $k(\lambda) = 2$. Thus (13.1) follows in all cases.

We note that (13.1) implies that for *any* eigenvalue $k(\lambda) \leqslant 2$. Also any eigenfunction can be expressed solely in terms of ϕ_1 and ϕ_2 (or, in a similar fashion, solely in terms of χ_1 and χ_2).

14. We require the following

LEMMA (14.1). *For all values of* λ

$$\int_a^b \phi_i(a \mid x, \lambda)\, \chi_j(b \mid x, \lambda)\, dx = P'_{ij}(\lambda) \quad (i, j = 1, 2), \tag{14.1}$$

where the prime denotes differentiation with respect to λ.

Proof. Applying Green's formula (5.1) to $\phi_i(x, \lambda+\rho)$ and $\chi_j(x, \lambda-\rho)$ where ρ is arbitrary but not zero, we get

$$\int_a^b \phi_i(x, \lambda+\rho)\, \chi_j(x, \lambda-\rho)\, dx = \frac{1}{2\rho}[P\{\phi_i(x, \lambda+\rho)\, \chi_j(x, \lambda-\rho)\}]_a^b$$

$$= \frac{P_{ij}(\lambda+\rho)-P_{ij}(\lambda-\rho)}{2\rho} + P_b\left\{\phi_i(x, \lambda+\rho)\frac{\chi_j(x, \lambda-\rho)-\chi_j(x, \lambda+\rho)}{2\rho}\right\} -$$

$$-P_a\left\{\frac{\phi_i(x, \lambda+\rho)-\phi_i(x, \lambda-\rho)}{2\rho}\chi_j(x, \lambda-\rho)\right\}.$$

Now let $\rho \to 0$, to give

$$\int_a^b \phi_i(x, \lambda)\, \chi_j(x, \lambda)\, dx = P'_{ij}(\lambda) - P_b\left\{\phi_i(x, \lambda)\frac{\partial \chi_j(x, \lambda)}{\partial \lambda}\right\} - P_a\left\{\frac{\partial \phi_i(x, \lambda)}{\partial \lambda}\chi_j(x, \lambda)\right\}$$
$$= P'_{ij}(\lambda)$$

since, by definition, $\phi_i(x, \lambda)$ and $\chi_j(x, \lambda)$ are independent of λ at $x = a$ and $x = b$ respectively.

15. In this section and the next I give some information about the residue of $\Phi(x, \lambda)$ at its poles.

Let $\lambda = \mu$ be an eigenvalue, i.e. $W(\mu) = 0$. We drop explicit reference to the fact that $\lambda = \mu$ and write P_{ij} for $P_{ij}(\mu)$, ϕ_1 or $\phi_1(x)$ for $\phi_1(a \mid x, \mu)$, etc. All these will be real-valued since μ is real.

Suppose firstly that $r(\mu) = 1$; we make no assumption about the order of the zero of $W(\lambda)$ at $\lambda = \mu$. One at least of the P_{ij} is not zero and without loss of generality we take $P_{22} \neq 0$. The other P_{ij} may be zero or otherwise. It is clear that

$$W' = P_{11}P'_{22} + P'_{11}P_{22} - P_{12}P'_{21} - P'_{12}P_{21},$$

where, as in all that follows, the prime denotes differentiation with respect to λ evaluated at $\lambda = \mu$.

Since $W = 0$, there is a linear relationship

$$\alpha_1 \phi_1 + \alpha_2 \phi_2 = \beta_1 \chi_1 + \beta_2 \chi_2 \quad (x \in [a,b]). \tag{15.1}$$

Using the self-adjointness conditions (7.1) on ϕ_i and χ_i we now have

$$\alpha_1 P_{12} + \alpha_2 P_{22} = 0, \qquad \beta_1 P_{21} + \beta_2 P_{22} = 0, \tag{15.2}$$

and from this and the hypothesis $P_{22} \neq 0$ it follows that

$$\alpha_1 \neq 0, \qquad \beta_1 \neq 0.$$

Thus (15.1) becomes

$$P_{22}\chi_1 - P_{21}\chi_2 = k\{P_{22}\phi_1 - P_{12}\phi_2\} \quad (x \in [a,b]), \tag{15.3}$$

where $k = \alpha_1/\beta_1$, so that k is neither zero nor infinite [compare Titchmarsh (1) 8].

From Lemma (14.1) we have

$$\int_a^b \{P_{22}\phi_1 - P_{12}\phi_2\}\{P_{22}\chi_1 - P_{21}\chi_2\}\, dx = W' P_{22},$$

and this combined with (15.3) gives

$$k \int_a^b \{P_{22}\phi_1 - P_{12}\phi_2\}^2\, dx = P_{22} W'.$$

Thus we have

(i) $\lambda = \mu$ is a *simple* zero of $W(\lambda)$;

(ii) $k[P_{22} W']^{-1} > 0$;

(iii) if we define

$$\psi(x,\mu) \equiv \left(\frac{k}{P_{22} W'}\right)^{\frac{1}{2}} (P_{22}\phi_1 - P_{12}\phi_2),$$

then $\psi(x,\mu)$ is a real-valued, normalized eigenfunction.

It is now clear that both $G(x,\xi;\lambda)$ and $\Phi(x,\lambda)$ have a simple pole at $\lambda = \mu$. It can be shown, but I omit the details, that the residue of $\Phi(x,\lambda)$ at $\lambda = \mu$ is

$$\psi(x,\mu) \int_a^b \psi(\xi,\mu) f(\xi)\, d\xi.$$

16. Now suppose that $\lambda = \mu$ is an eigenvalue for which $r(\mu) = 0$. In this case all the P_{ij} are zero, but we make no assumption about the order to which W or the P_{ij} vanish although it is clear that W possesses a zero of *at least* the second order.

We have, in this case,

$$W'' = 2(P'_{11}P'_{22}-P'_{12}P'_{21}),$$

where some of the P'_{ij} may vanish.

From (11.4) and (11.5) we have the following results valid for x in $[a,b]$:

$$\chi_1 = \alpha\phi_1+\beta\phi_2, \qquad \chi_2 = \gamma\phi_1+\delta\phi_2 \text{ (say)}, \tag{16.1}$$

where $\Delta = \alpha\delta-\beta\gamma \neq 0$.

Let
$$I_{ij} \equiv \int\limits_a^b \phi_i\phi_j\,dx \quad (i,j = 1, 2). \tag{16.2}$$

Since μ is real, $I_{ii} > 0$ $(i = 1, 2)$.

Multiplying (16.1) by the appropriate ϕ_i and integrating over $[a,b]$ we obtain, from Lemma (14.1)

$$\left.\begin{aligned} P'_{11} &= \alpha I_{11}+\beta I_{12}, & P'_{12} &= \gamma I_{11}+\delta I_{12} \\ P'_{21} &= \alpha I_{21}+\beta I_{22}, & P'_{22} &= \gamma I_{21}+\delta I_{22} \end{aligned}\right\}, \tag{16.3}$$

and by direct calculation these give

$$\tfrac{1}{2}W'' = P'_{11}P'_{22}-P'_{12}P'_{21} = \Delta(I_{11}I_{22}-I_{12}I_{21}).$$

The right-hand side of this last result is not zero since $\Delta \neq 0$ and

$$I_{11}I_{22}-I_{12}I_{21} > 0$$

by the Cauchy–Schwarz inequality and the linear independence of ϕ_1 and ϕ_2 over $[a,b]$. Thus we have the conclusion that

$$W'' \neq 0,$$

i.e. the zero of $W(\lambda)$ at $\lambda = \mu$ for this type of eigenvalue is *double*. From this and the fact that all the P_{ij} are zero, it follows that $G(x,\xi;\lambda)$ and $\Phi(x,\lambda)$ again have a *simple* pole at this type of eigenvalue.

To evaluate the residue of $\Phi(x,\lambda)$ in this case we use what is virtually the Schmidt process of orthogonalization. We *select* a multiple of $\phi_1(x,\mu)$ to be a normalized eigenfunction and then add and subtract the term

$$\psi_1(x,\mu)\int\limits_a^b \psi_1(\xi,\mu)f(\xi)\,d\xi \tag{16.4}$$

from the residue of $\Phi(x,\lambda)$ at $\lambda = \mu$, where $\psi_1(x,\mu)$ is *defined* by

$$\psi_1(x,\mu) \equiv \left(\frac{\Delta}{\delta P'_{11}-\beta P'_{12}}\right)^{\frac{1}{2}}\phi_1. \tag{16.5}$$

It is readily calculated from (16.3) that

$$\Delta I_{11} = \delta P'_{11}-\beta P'_{12},$$

so that $\psi_1(x,\mu)$ is a real-valued, normalized eigenfunction.

The remainder of the residue of Φ can then be shown to be

$$\psi_2(x,\mu)\int_a^b \psi_2(\xi,\mu)f(\xi)\,d\xi, \tag{16.6}$$

where
$$\psi_2(x,\mu) = \frac{(\alpha P'_{12}-\gamma P'_{11})\phi_1-(\delta P'_{11}-\beta P'_{12})\phi_2}{\{(\delta P'_{11}-\beta P'_{12})(P'_{11}P'_{22}-P'_{12}P'_{21})\}^{\frac{1}{2}}}. \tag{16.7}$$

It is readily shown from (16.1) that

$$(\delta P'_{11}-\beta P'_{12})(P'_{11}P'_{22}-P'_{12}P'_{21}) > 0,$$

so that $\psi_2(x,\mu)$ is a real-valued eigenfunction and linearly independent of $\psi_1(x,\mu)$.

We can now *verify*, from (16.3), that $\psi_2(x,\mu)$ is normalized and that

$$\int_a^b \psi_1(x,\mu)\psi_2(x,\mu)\,dx = 0.$$

Thus ψ_1 and ψ_2 form a pair of normal orthogonal eigenfunctions.

The complete residue of $\Phi(x,\lambda)$ at $\lambda=\mu$ for this type of eigenvalue is the sum of the terms in (16.4) and (16.6), where ψ_1 and ψ_2 are the eigenfunctions for $\lambda=\mu$.

17. The information of the last two sections gives

THEOREM (17.1) *With the notation of previous sections*

(*a*) *$W(\lambda)$ has at most a zero of the second order at any eigenvalue;*

(*b*) *$G(x,\xi;\lambda)$ and $\Phi(x,\lambda)$ have simple poles at all zeros of $W(\lambda)$;*

(*c*) *the residue of $\Phi(x,\lambda)$ at a pole $\lambda=\mu$ is of the form*

$$\psi(x,\mu)\int_a^b \psi(\xi,\mu)f(\xi)\,d\xi,$$

where
$$\int_a^b \{\psi(x,\mu)\}^2\,dx = 1,$$

or
$$\sum_{i=1}^{2}\psi_i(x,\mu)\int_a^b \psi_i(\xi,\mu)f(\xi)\,d\xi,$$

where
$$\int_a^b \psi_i(x,\mu)\psi_j(x,\mu)\,dx = \delta_{ij}$$

for, respectively, a simple or double zero of $W(\lambda)$. (The $\psi(x,\mu)$ are the eigenfunctions at $\lambda=\mu$.)

18. I have not attempted to discuss the extension of the asymptotic nature of solutions of $Ly=\lambda y$ for large λ. Results similar to those given

in Titchmarsh (**1**) Chapter I can be obtained to show that the B.V.P. of § 2 always possesses an infinity of eigenvalues and that the corresponding eigenfunctions form a complete orthogonal set over $[a, b]$. The question of equiconvergence theorems raises several interesting results which might be discussed in a later note.

Several examples of the type of B.V.P. in § 2 were discussed by the author in the thesis mentioned in § 1.

The method described above can be applied to differential equations of any even order and to boundary conditions of mixed type. It is also probable that they could be used to tackle the problem of a more general character discussed in Kamke (**9**) 236–43.

Results have also been obtained in the case when L is a singular differential operator: these are an extension of the theorems in Chapters II and III of Titchmarsh (**1**).

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